ADVANCED FACTORING

❑ INTRODUCTION

We can now factor lots of quadratic binomials ($4x^2 - 9$) and trinomials ($n^2 + 10n + 21$). Sorry to have to tell you this, but we're not done with factoring just yet. In this chapter, we learn how to factor expressions with the exponent 4 in them, expressions containing four terms, expressions containing GCFs you might never have seen before, and expressions that are the sum or difference of cubes.

❑ FACTORING QUARTICS

EXAMPLE 1: **Factor each quartic (4th degree) polynomial:**

A. $\boldsymbol{c^4 - 256}$

$= (c^2 + 16)(c^2 - 16)$ (difference of squares)

$= \boxed{(c^2 + 16)(c + 4)(c - 4)}$ (difference of squares again)

Note: $c^2 + 16$ cannot be factored any further.

B. $\boldsymbol{9a^4 - 37a^2 + 4}$

$= (9a^2 - 1)(a^2 - 4)$ (factor trinomial)

Now we notice that each factor is quadratic and is the difference of two squares. Therefore, each factor can be factored further to get a final answer consisting of four factors

$\boxed{(3a + 1)(3a - 1)(a + 2)(a - 2)}$

Homework

1. Factor each quartic polynomial:

a. $x^4 - 1$

b. $x^4 - x^2 - 6$

c. $n^4 - 10n^2 + 9$

d. $a^4 - 81$

e. $36w^4 - 25w^2 + 4$

f. $9x^4 - 34x^2 + 25$

g. $c^4 - 16$

h. $x^4 - 8x^2 - 9$

i. $x^4 - 3x^2 - 10$

j. $g^4 - 256$

k. $36u^4 - 85u^2 + 9$

l. $y^4 + 81$

❑ *THE GCF REVISITED*

EXAMPLE 2: **Factor: $(a + b)^2 + 4(a + b)$**

Solution: There are two terms in this expression: $(a + b)^2$ and $4(a + b)$. Notice that each of these two terms contains the same factor, namely $a + b$. In other words, the GCF of the two terms is $a + b$. Factoring out this GCF gives us the final factored form, a single term consisting of two factors:

$$(a + b)(a + b + 4)$$

The thing not to do in this kind of problem is to distribute the original expression; if you do, you'll be going in the wrong direction. Check it out:

$$(a + b)^2 + 4(a + b) = a^2 + 2ab + b^2 + 4a + 4b$$

Do you really want to try to factor that last expression?

So, when you see an expression, like $a + b$ in this problem, occurring multiple times in an expression, it's usually best to leave it intact. Also notice that we have converted a 2-termed expression into 1 term -- we have factored.

Alternate Method: Let's try a substitution method. We might be able to better see the essence of the problem if we replace $a + b$ with a simpler symbol -- for example, x will represent $a + b$. Then the original expression

$$(a + b)^2 + 4(a + b)$$

is transformed into

$$x^2 + 4x$$

The GCF in this form is clearly x, so we pull it out in front:

$$x(x + 4)$$

Now substitute in the reverse direction, to get $a + b$ back in the problem:

$$(a + b)(a + b + 4)$$ (the same answer as before)

<u>EXAMPLE 3:</u> **Factor:** $\boldsymbol{x^2(u - w) - 100(u - w)}$

<u>Solution:</u> The two given terms have a GCF of $u - w$. Factoring this GCF out gives

$$(u - w)(x^2 - 100)$$

But we're not done yet. The second factor is a difference of squares. Factoring that part gives us our final factorization:

$$\boxed{(u - w)(x + 10)(x - 10)}$$

EXAMPLE 4: **Factor:** $w^2(x+z) - 4w(x+z) + 3(x+z)$

Solution: Let's use substitution to make this expression appear a little less intimidating; we'll convert every occurrence of $x + z$ to the symbol A:

$$w^2A - 4wA + 3A$$

Pulling out the GCF of A, we get

$$A(w^2 - 4w + 3)$$

Factor the trinomial in the usual way:

$$A(w-3)(w-1)$$

Last, replace the A with its original definition of $x + z$:

$$\boxed{(x+z)(w-3)(w-1)}$$

Homework

2. Factor each expression:

a. $(x+y)^2 + 7(x+y)$

b. $(a-b)^2 - c(a-b)$

c. $x^2(c+d) + 5(c+d)$

d. $n^2(a-b) - 9(a-b)$

e. $x^2(a+4) + 5x(a+4) + 6(a+4)$

f. $y^2(m+n) + 7y(m+n)$

g. $2x^2(a+b) + 3x(a+b) - 5(a+b)$

h. $4x^2(w+z) - 9(w+z)$

i. $(u-w)^2 - 9(u-w)$

j. $n^2(a+b) - 9n(a+b)$

k. $(t+r)y^2 - 100(t+r)$

l. $3ax^2 - 20ax - 7a$

❑ GROUPING WITH FOUR TERMS

EXAMPLE 5: **Factor: $a^2 + ac + ab + bc$**

Solution: Group the first two terms and the last two terms:

$$(a^2 + ac) + (ab + bc)$$

Now factor each pair of grouped terms separately (using the GCF in each pair) :

$$a(a + c) + b(a + c)$$ (notice that the GCF is $a + c$)

Even though we've grouped and factored, we can't be done because there are still two terms, and we need one term in the final answer to a factoring question. So we continue -- using our knowledge of the previous section -- and factor out the GCF, which is $a + c$:

$$\boxed{(a + c)(a + b)}$$

By the commutative property of multiplication ($xy = yx$), the final answer could also be written $(a + b)(a + c)$. Also, to check our answer, just double distribute the answer and you should get the original expression.

EXAMPLE 6: **Factor: $x^3 - 7x^2 - 9x + 63$**

Solution: Group the first two terms and the last two terms:

$$(x^3 - 7x^2) + (-9x + 63)$$

Now factor the GCF in each pair of grouped terms. The first GCF is obvious: x^2. Choosing the GCF in the second grouping is a little trickier -- should we choose 9 or -9? Ultimately, it's a trial-and-error process. Watch what happens if we choose -9 for the GCF:

$x^2(x - 7) - 9(x - 7)$ (check the signs carefully)

We now see two terms whose GCF is $x - 7$. Pull it to the front:

$(x - 7)(x^2 - 9)$

All this, and we're still not done. The second factor is the difference of two squares -- now we're done:

$$(x - 7)(x + 3)(x - 3)$$

EXAMPLE 7: **Factor:** $\boldsymbol{ab + cd + ad + bc}$

Solution: Group the first two terms and the last two terms (after all, this technique worked quite well in the previous two examples):

$(ab + cd) + (ad + bc)$

We're stuck; there's no way to factor either pair of terms (the GCF = 1 in each case), so let's swap the two middle terms of the original problem and again group in pairs:

$(ab + ad) + (cd + bc)$

Pull out the GCF from each set of parentheses:

$a(b + d) + c(d + b)$

Do we have a common factor in these two terms? Well, does $b + d = d + b$? Since addition is commutative, of course they are equal. So the GCF is $b + d$, and when we pull it out in front, we're done:

$$(b + d)(a + c)$$

<u>EXAMPLE 8:</u> **Factor: $2ax - bx - 2ay + by$**

<u>Solution:</u> Group in pairs, as usual:

$$(2ax - bx) + (-2ay + by)$$

Pull out the GCF in each grouping:

$$x(2a - b) + y(-2a + b)$$

<u>Problem:</u> There's no common factor; however, the factors $2a - b$ and $-2a + b$ are <u>opposites</u> of each other, and that gives us a clue. Let's go back to our first step and factor out $-y$ rather than y:

$$x(2a - b) - y(2a - b)$$ (distribute to make sure we're right)

Now we see a good GCF, so we pull it out in front, and we're done:

$$\boxed{(2a - b)(x - y)}$$ [Check by multiplying out]

Homework

3. Factor each expression:

a. $xw + xz + wy + yz$

b. $a^2 + ac + ab + bc$

c. $x^3 - 4x^2 + 3x - 12$

d. $n^3 - n^2 - 5n + 5$

e. $x^3 + x^2 - 9x - 9$

f. $ac - bd + bc - ad$

g. $xw + yz - xz - wy$

h. $2ac - 2ad + bc - bd$

i. $6xw - yz + 3xz - 2wy$

j. $hj - j^2 - hk + jk$

k. $ax + ay - bx - by$

l. $x^3 - 2x^2 - 25x + 50$

m. $xw + 2wy - xz - 2yz$

n. $a^3 - a^2 - 5a + 5$

o. $4tw - 2tx + 2w^2 - wx$

p. $6x^3 + 2x^2 - 9x - 3$

q. Not factorable

r. $6a^3 - 15a^2 + 10a - 25$

❑ *MORE GROUPING AND SUBSTITUTION PROBLEMS*

EXAMPLE 9: **Factor:** $(w + z)^2 - a^2$

Solution: After some practice, you might not need a substitution for this kind of problem, but we'll use one for this problem. Let $n = w + z$. The starting problem then becomes

$$n^2 - a^2$$

This is just a standard difference of squares:

$$(n + a)(n - a)$$

Now substitute in the other direction:

$$\boxed{(w + z + a)(w + z - a)}$$

EXAMPLE 10: **Factor:** $x^2 + 6x + 9 - y^2$

Solution: Grouping in pairs has worked quite well so far, so let's try it again:

$$(x^2 + 6x) + (9 - y^2)$$

We see that the first pair of terms has a nice GCF of x, and the second is the difference of squares:

$$x(x + 6) + (3 + y)(3 - y)$$

Good try, but there's no common factor in these two terms. In fact, no grouping into pairs will result in a common factor -- a dead end. Let's go back to the original problem and regroup so that the first three terms are together:

$$(x^2 + 6x + 9) - y^2$$

The first set of three terms is a perfect square trinomial, and factors into the square of a binomial:

$$(x+3)^2 - y^2$$

leaving us with another difference of squares (just like the previous example), which factors to

$$\boxed{(x+3+y)(x+3-y)}$$

Homework

4. Factor each expression:

a. $(x+y)^2 - z^2$

b. $(a-b)^2 - c^2$

c. $x^2 + 4x + 4 - y^2$

d. $n^2 - 6n + 9 - Q^2$

e. $(u+w)^2 - T^2$

f. $y^2 + 10y + 25 - x^2$

g. $a^2 + 2ab + b^2 - c^2$

h. $w^2 - 2wy + y^2 - 49$

i. $4x^2 + 4x + 1 - t^2$

j. $9x^2 - 12x + 4 - y^2$

❑ *FACTORING CUBICS USING THE GCF*

<u>EXAMPLE 11</u>: **Factor each cubic (3rd degree) polynomial:**

A. $\mathbf{5q^3 + 10q^2 + 5q}$

This is not as bad as it looks, <u>if</u> we remember to start with the GCF:

	$5q^3 + 10q^2 + 5q$	(the polynomial to factor)
=	$5q(q^2 + 2q + 1)$	(factor out $5q$, the GCF)

$=$	$5q(q + 1)(q + 1)$	(factor the trinomial)
$=$	$\boxed{5q(q + 1)^2}$	(write it more simply)

B. $\mathbf{4x^3 - x}$

$=$	$x(4x^2 - 1)$	(factor out x, the GCF)
$=$	$\boxed{x(2x + 1)(2x - 1)}$	(difference of squares)

Homework

5. Factor each cubic polynomial:

a. $x^3 - x$

b. $2n^3 + 6n^2 + 4n$

c. $10a^3 - 5a^2 - 5a$

d. $7y^3 + 70y^2 + 175y$

e. $36w^3 - 9w$

f. $24z^3 - 20z^2 - 24z$

Practice Problems

6. Factor each expression:

a. $10ax^4 - 160a$

b. $Z^2(P - Q) - 144(P - Q)$

c. $50x^3 - 75x^2 - 2x + 3$

d. $12ac - 10bd + 8bc - 15ad$

e. $a^2 - 2ab + b^2 - c^2$

f. $x^2 + 2xy + y^2 - 144$

g. $x^4 - 34x^2 + 225$

h. $x^4 - 8x^2 - 9$

i. $x^3 - 7x^2 + 9x - 63$

j. $n^3 + 3n^2 - 16n - 48$

k. $(a+b)^2 - 5(a+b) + 6$

l. $(x-y)^2 + 7(x-y) + 6$

m. $(a-b)^2 + 6(a-b) - 16$

n. $hm - hn + km - kn$

Solutions

1.
a. $(x^2+1)(x+1)(x-1)$
b. $(x^2+2)(x^2-3)$
c. $(n+1)(n-1)(n+3)(n-3)$
d. $(a^2+9)(a+3)(a-3)$
e. $(2w+1)(2w-1)(3w+2)(3w-2)$
f. $(x+1)(x-1)(3x+5)(3x-5)$
g. $(c^2+4)(c+2)(c-2)$
h. $(x^2+1)(x+3)(x-3)$
i. $(x^2+2)(x^2-5)$
j. $(g^2+16)(g+4)(g-4)$
k. $(2u+3)(2u-3)(3u+1)(3u-1)$
l. Not factorable

2.
a. $(x+y)(x+y+7)$
b. $(a-b)(a-b-c)$
c. $(c+d)(x^2+5)$
d. $(a-b)(n+3)(n-3)$
e. $(a+4)(x+3)(x+2)$
f. $y(m+n)(y+7)$
g. $(a+b)(2x+5)(x-1)$
h. $(w+z)(2x+3)(2x-3)$
i. $(u-w)(u-w-9)$
j. $n(a+b)(n-9)$
k. $(t+r)(y+10)(y-10)$
l. $a(3x+1)(x-7)$

3.
a. $(x+y)(w+z)$
b. $(a+b)(a+c)$
c. $(x^2+3)(x-4)$
d. $(n^2-5)(n-1)$
e. $(x+1)(x+3)(x-3)$
f. $(a+b)(c-d)$
g. $(x-y)(w-z)$
h. $(2a+b)(c-d)$
i. $(3x-y)(2w+z)$
j. $(h-j)(j-k)$
k. $(a-b)(x+y)$
l. $(x-2)(x+5)(x-5)$
m. $(x+2y)(w-z)$
n. $(a^2-5)(a-1)$
o. $(2t+w)(2w-x)$
p. $(2x^2-3)(3x+1)$
q. Not factorable
r. $(3a^2+5)(2a-5)$

4.
a. $(x+y+z)(x+y-z)$
b. $(a-b+c)(a-b-c)$
c. $(x+2+y)(x+2-y)$
d. $(n-3+Q)(n-3-Q)$
e. $(u+w+T)(u+w-T)$
f. $(y+5+x)(y+5-x)$
g. $(a+b+c)(a+b-c)$
h. $(w-y+7)(w-y-7)$

i. $(2x + 1 + t)(2x + 1 - t)$
j. $(3x - 2 + y)(3x - 2 - y)$

5.
a. $x(x + 1)(x - 1)$
b. $2n(n + 1)(n + 2)$
c. $5a(2a + 1)(a - 1)$
d. $7y(y + 5)^2$
e. $9w(2w + 1)(2w - 1)$
f. $4z(3z + 2)(2z - 3)$

6.
a. $10a(x^2 + 4)(x + 2)(x - 2)$
b. $(P - Q)(Z + 12)(Z - 12)$
c. $(2x - 3)(5x + 1)(5x - 1)$
d. $(3a + 2b)(4c - 5d)$
e. $(a - b + c)(a - b - c)$
f. $(x + y + 12)(x + y - 12)$
g. $(x + 5)(x - 5)(x + 3)(x - 3)$
h. $(x^2 + 1)(x + 3)(x - 3)$
i. $(x^2 + 9)(x - 7)$
j. $(n + 4)(n - 4)(n + 3)$
k. $(a + b - 3)(a + b - 2)$
l. $(x - y + 6)(x - y + 1)$
m. $(a - b + 8)(a - b - 2)$
n. $(m - n)(h + k)$

"A college degree is not a sign that one is a finished product, but an indication a person is prepared for life."

Reverend Edward A. Malloy, *Monk's Reflections*