FACTORING: COMPLETE

❑ INTRODUCTION

Consider the task of factoring $8x^2 + 12x$. Even though 2 is a common factor, and even though x is a common factor, neither of them is the GCF, the greatest common factor. In fact, even their product, $2x$, is not the GCF. The fact is, $4x$ is the GCF. This is the quantity which gets pulled out in front, using the distributive property, giving us the most *complete* (that is, the best) *factorization*:

$$8x^2 + 12x = 4x(2x + 3)$$

❑ THE CONCEPT OF COMPLETE FACTORING

EXAMPLE 1: **Factor completely: $10x^2 + 50x + 60$**

Solution: Look at the 10. Its factor pairs are 1 and 10, or 2 and 5. Now take a look at the 60. It's downright scary to consider all the pairs of factors of that number. But watch what happens if we deal with the GCF first, and then worry about the rest later.

The variable x is not common to all three terms, so we'll ignore it. But each of the three terms does contain a factor of 10 (since 10 divides evenly into each of the numbers 10, 50, and 60). Thus,

	$10x^2 + 50x + 60$	(the given expression)
=	$10(x^2 + 5x + 6)$	(pull out the GCF of 10)
=	$10(x + 3)(x + 2)$	(factor the quadratic)

Not so difficult, after all. Therefore, the complete factorization of $10x^2 + 50x + 60$ is

$$\boxed{10(x + 3)(x + 2)}$$

The key to complete factoring
is to FIRST pull out the GCF!

Homework

1. Factor each expression completely:

a. $7x^2 - 35x + 42$
b. $10n^2 - 10$
c. $5a^2 - 30a + 45$
d. $50u^2 - 25u - 25$
e. $7w^2 - 700$
f. $9n^2 + 9$
g. $5y^2 - 125$
h. $3x^2 + 15x + 12$
i. $14x^2 - 7x - 7$
j. $13t^2 + 117$
k. $48z^2 - 28z + 4$
l. $24a^2 - 120a + 150$

❑ *REDUCING FRACTIONS*

EXAMPLE 2: **Reduce to lowest terms:** $\dfrac{5a^2 - 45}{a^2 + 6a + 9}$

Solution: We need to factor the numerator and the denominator. If we then see any common factors, we can divide them out.

$$\frac{5a^2-45}{a^2+6a+9} = \frac{5(a^2-9)}{a^2+6a+9} = \frac{5(a+3)(a-3)}{(a+3)(a+3)} = \frac{5\cancel{(a+3)}(a-3)}{\cancel{(a+3)}(a+3)}$$

Notice that factoring the numerator required two steps: pulling out the GCF of 5, followed by factoring the $a^2 - 9$. If we hadn't factored out the 5, we would never have been able to divide out anything, and we would have reached the false conclusion that the fraction is not reducible. So we can write the reduced fraction as $\frac{5(a-3)}{a+3}$. But, assuming it's not too much work, it's customary to remove parentheses. Thus, the final answer (after distributing the 5 to the $a - 3$) is

$$\boxed{\frac{5a-15}{a+3}}$$

Homework

2. Reduce each fraction to lowest terms:

a. $\dfrac{n^2-4}{n^2-4n+4}$

b. $\dfrac{2x^2+8x+6}{6x^2+18x+12}$

c. $\dfrac{x^2-4x+1}{x^2-4x+1}$

d. $\dfrac{10y^2-30y+20}{5y^2-15y+10}$

e. $\dfrac{2x^2-2}{4x-4}$

f. $\dfrac{3n^2-3n-90}{3n^2+30n+75}$

g. $\dfrac{14x+98}{21x^2-63x-1470}$

h. $\dfrac{5a^2-30a-135}{10a^2-60a-270}$

❑ ADDITIONAL QUADRATIC EQUATIONS

Now we'll combine the GCF method of factoring with the ideas from this chapter to solve more quadratic equations. The following example should convince you that factoring out a simple number first makes the rest of the factoring -- and thus the solving of the equation -- vastly easier.

EXAMPLE 3: **Solve for k: $16k^2 = 40k + 24$**

Solution: Solving a quadratic equation by factoring requires that we make one side of the equation zero. To this end, we will first bring the $40k$ and the 24 to the left side, factor in two steps, divide each side by the greatest common factor, set each factor to 0, and then solve each resulting equation. (That's a lot of steps!)

$16k^2 = 40k + 24$ (the original equation)

$\Rightarrow 16k^2 - 40k - 24 = 0$ (subtract $40k$ and 24)

$\Rightarrow 8(2k^2 - 5k - 3) = 0$ (factor out 8, the GCF)

$\Rightarrow \dfrac{8(2k^2 - 5k - 3)}{8} = \dfrac{0}{8}$ (divide each side by 8)

$\Rightarrow 2k^2 - 5k - 3 = 0$ (simplify)

$\Rightarrow (2k + 1)(k - 3) = 0$ (factor)

$\Rightarrow 2k + 1 = 0 \quad \text{or} \quad k - 3 = 0$ (set each factor to 0)

$\Rightarrow \boxed{k = -\frac{1}{2} \quad \text{or} \quad k = 3}$ (solve each linear equation)

This was a lot of work -- you might be thinking that the Quadratic Formula (assuming you've studied it) would have been easier to use than factoring, and you're probably right! Generally, you can use whichever method you prefer (assuming it works), but pay attention to the method your teacher may require on an exam.

❑ CAREFUL ! ! !

Do you see the step in the preceding example where we divided both sides of the equation by 8? This was legal because we did the same thing to both sides of the equation, and we did NOT divide by zero. Do not ever fall into the trap of dividing each side of an equation by something with the variable in it -- you may accidentally be dividing by zero, and thus you may lose a solution to the equation.

To illustrate this, the correct way to solve the quadratic equation $x^2 + x = 0$ is to do the following:

$$x^2 + x = 0$$

$$\Rightarrow \quad x(x + 1) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x + 1 = 0$$

$$\Rightarrow \quad \underline{x = 0} \text{ or } \underline{x = -1}$$

That is, we have two solutions: 0 and -1 -- no doubt about it.

Check: $x = 0:\quad \mathbf{0}^2 + \mathbf{0} = 0 + 0 = 0$ ✓

$x = -1:\quad (\mathbf{-1})^2 + (\mathbf{-1}) = 1 + (-1) = 1 - 1 = 0$ ✓

Now let's do it the wrong way, by dividing by the variable:

$$x^2 + x = 0 \Rightarrow x(x+1) = 0 \Rightarrow \frac{\cancel{x}(x+1)}{\cancel{x}} = \frac{0}{x} \Rightarrow x + 1 = 0$$

$\Rightarrow$ $x = -1$, which is merely one of the two solutions. That is, *we lost a solution when we divided by the variable.* Since the purpose of algebra is to obtain solutions -- not throw them away -- we see that dividing by the variable was a really bad idea.

Homework

3. Solve each quadratic equation:

a. $7x^2 - 35x + 42 = 0$

b. $10n^2 = 10$

c. $5a^2 + 45 = 30a$

d. $50u^2 = 25u + 25$

e. $7w^2 - 700 = 0$

f. $180z^2 - 30z - 60 = 0$

g. $4x^2 + 4x - 24 = 0$

h. $16x^2 = 6 - 4x$

i. $10x^2 - 490 = 0$

j. $75w^2 + 48 = 120w$

Review Problems

4. Factor completely: $30q^2 + 68q + 30$

5. Reduce: $\dfrac{18x+18}{14x^2+42x+28}$

6. Solve for x by factoring: $30x^2 = 190x + 140$

Solutions

1. a. $7(x-3)(x-2)$ b. $10(n+1)(n-1)$ c. $5(a-3)^2$

d. $25(2u+1)(u-1)$ e. $7(w+10)(w-10)$ f. $9(n^2+1)$

g. $5(y+5)(y-5)$ h. $3(x+1)(x+4)$ i. $7(2x+1)(x-1)$

j. $13(t^2+9)$ k. $4(4z-1)(3z-1)$ l. $6(2a-5)^2$

2. a. $\frac{n+2}{n-2}$ b. $\frac{x+3}{3x+6}$ c. 1 d. 2

e. $\frac{x+1}{2}$ f. $\frac{n-6}{n+5}$ g. $\frac{2}{3x-30}$ h. $\frac{1}{2}$

3. a. 2, 3 b. ± 1 c. 3 d. $1, -\frac{1}{2}$ e. ± 10

f. $\frac{2}{3}, -\frac{1}{2}$ g. 2, -3 h. $\frac{1}{2}, -\frac{3}{4}$ i. ± 7 j. $\frac{4}{5}$

4. $2(5x+3)(3x+5)$ **5.** $\frac{9}{7x+14}$ **6.** $7, -\frac{2}{3}$

❑ *To ∞ and Beyond*

A. Factor: $x^2 - 363x + 21{,}600$

Hint: Use the Quadratic Formula (and your calculator) to solve the equation $x^2 - 363x + 21{,}600 = 0$, and then use the solutions to write the factorization.

B. Solve for x: $x^4 - 13x^2 + 36 = 0$

Hint: Factoring will take two steps, and there will be four solutions.

– Definition of ***happiness*** by John F. Kennedy (1917-1963)

“The full use of your powers along lines of excellence.”