# FACTORING THE SUM AND DIFFERENCE OF CUBES

## ❑ *INTRODUCTION*

We've learned that we can factor the *difference of squares* $x^2 - y^2$ into $(x + y)(x - y)$. We've also determined that the *sum of squares* $x^2 + y^2$ cannot be factored. Now we're about to show that the ***difference of cubes*** $x^3 - y^3$ can also be factored -- and perhaps surprisingly -- even the ***sum of cubes*** $x^3 + y^3$ can be factored. We begin with a discussion of division, remainders, and factors.

Is 6 a factor of 161? No -- divide 161 by 6 and you'll get 26 remainder **5**. Since the remainder is not zero, 6 is not a factor of 161. In other words, 6 does not go into 161 "evenly."

Is 7 a factor of 161? Yes -- divide 161 by 7 and you'll get 23, remainder **0**. Thus, 7 divides into 161 exactly 23 times. And therefore, $161 = 7 \times 23$. We have factored 161 into $7 \times 23$ by showing that the factor 7 divides into 161 without remainder. These observations are the key to factoring the sum and difference of cubes.

### Perfect Cubes

We know that $2^3 = 8$. Since the cube of 2 is 8, we say that 8 is a ***perfect cube***. Here are some more examples of perfect cubes:

125 is a perfect cube because it's the cube of 5.

1 is a perfect cube because it's the cube of 1.

$-27$ is a perfect cube because it's the cube of $-3$.

0 is a perfect cube, but who cares?

$x^3$ is a perfect cube because it's the cube of $x$.

$27y^3$ is a perfect cube because it's the cube of $3y$.

$8n^6$ is a perfect cube because it's the cube of $2n^2$.

$64z^{12}$ is a perfect cube because it's the cube of $4z^4$.

## Homework

**1.** a. $64m^3$ is a perfect cube because it's the cube of _____.

b. $216n^3$ is a perfect cube because it's the cube of _____.

c. $27A^6$ is a perfect cube because it's the cube of _____.

d. _____ is a perfect cube because it's the cube of $7z^2$.

e. _____ is a perfect cube because it's the cube of $-3a^3$.

## ❑ *FACTORING A SUM OF CUBES*

We're now ready to try to factor a sum of cubes; for example, what is the factorization of

$$x^3 + 8\ ?$$

To answer this question, we should try to divide $x^3 + 8$ by something that goes into it evenly; that is, divide $x^3 + 8$ by something that will leave a remainder of zero. But what should we divide by? Since both terms of $x^3 + 8$ are perfect cubes, let's divide it by the binomial $x + 2$, since these two terms are the cube roots of $x^3$ and 8. Maybe this will work and maybe it won't, but we've got to try something.

$$
\begin{array}{r|l}
 & x^2 - 2x + 4 \\
\hline
x+2 & x^3 + 0x^2 + 0x + 8 \\
 & \underline{x^3 + 2x^2} \\
 & \quad -2x^2 + 0x \\
 & \quad \underline{-2x^2 - 4x} \\
 & \qquad\quad 4x + 8 \\
 & \qquad\quad \underline{4x + 8} \\
 & \qquad\qquad 0
\end{array}
$$

Here's the division of $x^3 + 8$ by $x + 2$. Note that the dividend has two zeros placed in it to account for the missing terms.

Also note that the remainder is 0. This means that $x + 2$ is a factor of $x^3 + 8$ and therefore, that $x^2 - 2x + 4$ is the other factor.

Now we write the results of our long division in the form of a multiplication problem, giving us the factorization of $x^3 + 8$:

$$\boxed{x^3 + 8 = (x+2)(x^2 - 2x + 4)}$$

## ❑ *Factoring a Difference of Cubes*

For our difference of cubes, let's try to factor $n^3 - 27$. What do you think one of the factors will be? Consider the binomial consisting of the individual cube roots of $n^3$ and $-27$, namely $n - 3$. This time it's your turn to carry out the long division. Here's what you should end up with:

$$
\begin{array}{r|l}
 & n^2 + 3n + 9 \\
\hline
n-3 & n^3 + 0n^2 + 0n - 27
\end{array}
$$

We now have our factorization:

$$\boxed{n^3 - 27 = (n-3)(n^2 + 3n + 9)}$$

EXAMPLE 1:

A. Factor: $N^3 - 1$. Divide $N^3 - 1$ by $N - 1$ and you should get the factorization $\mathbf{N^3 - 1 = (N - 1)(N^2 + N + 1)}$.

B. Factor: $8p^3 + 27$. Divide $8p^3 + 27$ by $2p + 3$. It should divide evenly, thus giving $\mathbf{8p^3 + 27 = (2p + 3)(4p^2 - 6p + 9)}$.

C. Factor: $(a + b)^3 - 125$. This is tricky, and it will be much easier to perform the long division if we make a substitution first. If we let $x = a + b$, then the expression to factor becomes $x^3 - 125$. The appropriate quantity to divide this by would be $x - 5$. When the long division is finished, the quotient is $x^2 + 5x + 25$ with remainder 0. We therefore get the factorization

$$x^3 - 125 = (x - 5)(x^2 + 5x + 25)$$

But the original problem didn't have any $x$'s in it. So we need to substitute back the other way -- converting each $x$ back into $a + b$, we get the factorization

$$(a + b)^3 - 125 = \big((a + b) - 5\big)\big((a + b)^2 + 5(a + b) + 25\big),$$

which can be simplified to the final answer of

$$\mathbf{(a + b)^3 - 125 = (a + b - 5)(a^2 + 2ab + b^2 + 5a + 5b + 25)}.$$

# Homework

2. In the discussion above, we arrived at the following factorizations:

   a. $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

   b. $n^3 - 27 = (n - 3)(n^2 + 3n + 9)$

   Verify each result by simplifying the right side of the statement so that it becomes the left side.

3. Factor each expression:

a. $x^3 - 8$ b. $n^3 + 27$ c. $z^3 + 1$

d. $8x^3 - 27$ e. $27y^3 + 125$ f. $64a^3 - 1$

4. Factor each expression:

a. $(x + y)^3 + 8$ b. $(a - b)^3 - 27$ c. $(p + q)^3 + 1$

5. Factor $x^5 + 1$. Hint: Divide by $x + 1$.

6. Factor $A^5 - 32$. Hint: Divide by $A - 2$.

7. Factor each expression:

a. $w^5 + 1$ b. $c^5 - 1$ c. $y^5 - 32$

d. $z^5 + 32$ e. $n^5 + 243$ f. $m^5 - 243$

8. Factor each expression:

a. $x^7 - 1$ b. $y^7 + 1$ c. $u^7 - 128$ d. $z^7 + 128$

# Solutions

**1.** a. $4m$ b. $6n$ c. $3A^2$ d. $343z^6$ e. $-27a^9$

**2.** a. $(x + 2)(x^2 - 2x + 4) = x^3 - 2x^2 + 4x + 2x^2 - 4x + 8 = x^3 + 8$ ✓

b. You try it.

**3.** a. $(x - 2)(x^2 + 2x + 4)$ b. $(n + 3)(n^2 - 3n + 9)$

c. $(z + 1)(z^2 - z + 1)$ d. $(2x - 3)(4x^2 + 6x + 9)$

e. $(3y + 5)(9y^2 - 15y + 25)$ f. $(4a - 1)(16a^2 + 4a + 1)$

**4.** a. $(x + y + 2)(x^2 + 2xy + y^2 - 2x - 2y + 4)$

b. $(a - b - 3)(a^2 - 2ab + b^2 + 3a - 3b + 9)$

c. $(p + q + 1)(p^2 + 2pq + q^2 - p - q + 1)$

**5.** $(x + 1)(x^4 - x^3 + x^2 - x + 1)$

**6.** $(A - 2)(A^4 + 2A^3 + 4A^2 + 8A + 16)$

**7.** a. $(w + 1)(w^4 - w^3 + w^2 - w + 1)$

b. $(c - 1)(c^4 + c^3 + c^2 + c + 1)$

c. $(y - 2)(y^4 + 2y^3 + 4y^2 + 8y + 16)$

d. $(z + 2)(z^4 - 2z^3 + 4z^2 - 8z + 16)$

e. $(n + 3)(n^4 - 3n^3 + 9n^2 - 27n + 81)$

f. $(m - 3)(m^4 + 3m^3 + 9m^2 + 27m + 81)$

**8.** a. $(x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

b. $(y + 1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1)$

c. $(u - 2)(u^6 + 2u^5 + 4u^4 + 8u^3 + 16u^2 + 32u + 64)$

d. $(z + 2)(z^6 - 2z^5 + 4z^4 - 8z^3 + 16z^2 - 32z + 64)$

**9.** a. $10a(x^2 + 4)(x + 2)(x - 2)$

b. $(P - Q)(Z + 12)(Z - 12)$

c. $(2x - 3)(5x + 1)(5x - 1)$

d. $(3a + 2b)(4c - 5d)$

e. $(a - b + c)(a - b - c)$

f. $(x + y + 12)(x + y - 12)$

g. $(x + 5)(x - 5)(x + 3)(x - 3)$

h. $(x^2 + 1)(x + 3)(x - 3)$

i. $(x^2 + 9)(x - 7)$

j. $(n + 4)(n - 4)(n + 3)$

k. $(a + b - 3)(a + b - 2)$

l. $(x - y + 6)(x - y + 1)$

m. $(a - b + 8)(a - b - 2)$

n. $(m - n)(h + k)$

**10**. a. $(n + 4)(n^2 - 4n + 16)$ b. $(a - 5)(a^2 + 5a + 25)$

c. $(2T - 3)(4T^2 + 6T + 9)$ d. $(3x + 1)(9x^2 - 3x + 1)$

*"Keep away from people who try to belittle your ambitions. Small people always do that, but the really great make you feel that you, too, can become great."*

**Mark Twain**