FRACTIONS: AN INTRODUCTION

❑ INTRODUCTION

We're about to embark on a study of fractions. If you've understood arithmetic fractions before, you'll find that algebraic fractions follow the same set of rules. If fractions are still a mystery, let's make this the semester that you master them.

First, no matter what a fraction is used for, or no matter how complicated it looks, a fraction ultimately represents DIVISION. For example, $\frac{6}{3}$ is a fraction, but it is also the division problem $6 \div 3$, which is why we write

$$\frac{6}{3} = 2$$

The fraction $\frac{1}{4}$ is a division problem (it equals 0.25) even if we never actually carry out the division.

The top of a fraction is called the **numerator**; the bottom is called the **denominator**.

The **reciprocal** of a fraction is obtained by swapping the numerator and denominator; it's created by "inverting" the fraction. As examples, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, and the reciprocal of x is $\frac{1}{x}$. The only number which does not possess a reciprocal is 0 -- if 0 had a reciprocal, it would have to be $\frac{1}{0}$, but we've learned that this fraction is undefined. Last, an interesting property of reciprocals is that the product of any number and its reciprocal is always 1. For example, $\frac{2}{3} \times \frac{3}{2} = \frac{6}{6} = 1$.

Homework

1. Convert each fraction to a whole number (if possible):

   a. $\frac{10}{2}$ b. $\frac{99}{9}$ c. $\frac{18}{1}$ d. $\frac{34}{17}$ e. $\frac{0}{75}$ f. $\frac{3}{0}$

2. Convert each fraction to a decimal (if possible):

   a. $\frac{1}{8}$ b. $\frac{3}{16}$ c. $\frac{1}{32}$ d. $\frac{43}{10}$ e. $\frac{9}{5}$ f. $\frac{14}{0}$

3. When converting the fraction $\frac{\text{numerator}}{\text{denominator}}$ to a long-division problem, which of the following is the proper setup?

   $\text{numerator}\overline{)\,\text{denominator}}$ $\text{denominator}\overline{)\,\text{numerator}}$

4. Find the **reciprocal**:

   a. $\frac{3}{2}$ b. $-\frac{4}{5}$ c. $-\frac{8}{7}$ d. 7 e. -14

   f. $-\frac{1}{7}$ g. a h. $\frac{1}{x}$ i. $\frac{x}{y}$ j. $-\frac{y}{x}$

   k. $-T$ l. $-\frac{1}{w}$ m. 1 n. -1 o. 0

5. a. What is the reciprocal of $-\frac{7}{19}$?

   b. What is the reciprocal of $-\frac{19}{7}$?

   c. Prove that $-\frac{17}{9}$ and $-\frac{9}{17}$ are reciprocals of each other by calculating their product.

❑ *ADDING AND SUBTRACTING FRACTIONS WITH THE SAME DENOMINATOR*

$$\frac{2}{9}+\frac{5}{9} = \frac{2+5}{9} = \frac{7}{9} \qquad \Rightarrow \qquad \frac{a}{b}+\frac{c}{b} = \frac{a+c}{b}$$

To **add** fractions with the same (common) denominator

i) the numerator of the answer is the sum of the two numerators

ii) the denominator of the answer is the common denominator

$$\frac{5}{11}-\frac{3}{11} = \frac{5-3}{11} = \frac{2}{11}$$

$$\frac{2}{13}-\frac{8}{13} = \frac{-6}{13} = -\frac{6}{13} \qquad \Rightarrow \qquad \frac{x}{y}-\frac{z}{y} = \frac{x-z}{y}$$

To **subtract** fractions with the same (common) denominator

i) the numerator of the answer is the difference of the two numerators

ii) the denominator of the answer is the common denominator

Homework

6. Add or subtract the fractions:

a. $\frac{7}{10}+\frac{2}{10}$ b. $\frac{43}{101}+\frac{44}{101}$ c. $\frac{7}{8}-\frac{3}{8}$ d. $\frac{1}{5}-\frac{3}{5}$

e. $\frac{x}{y}+\frac{z}{y}$ f. $\frac{m}{u}-\frac{q}{u}$ g. $\frac{c}{a}+\frac{d}{a}$ h. $\frac{t}{w}-\frac{z}{w}$

i. $\frac{b}{c}+\frac{7}{c}$ j. $\frac{9}{Q}-\frac{7}{Q}$ k. $\frac{5}{R}-\frac{10}{R}$ l. $\frac{1}{3a}+\frac{7}{3a}$

m. $\frac{a}{bc}-\frac{d}{bc}$ n. $\frac{a}{x^2}+\frac{b}{x^2}$ o. $\frac{3}{10x}+\frac{4}{10x}$ p. $\frac{u}{a^3}-\frac{b}{a^3}$

❑ ADDING AND SUBTRACTING FRACTIONS WITH DIFFERENT DENOMINATORS

$$\begin{aligned} & \frac{2}{3}+\frac{5}{7} \\ = \quad & \frac{2\bullet}{3\bullet}\ \ \frac{\bullet}{\bullet} \\ = \quad & \frac{14}{21}+\frac{15}{21} \\ = \quad & \frac{29}{21} \end{aligned}$$

$$\begin{aligned} & \frac{a}{b}+\frac{c}{d} \\ = \quad & \frac{a\bullet}{b\bullet}\ \ \frac{\bullet}{\bullet} \\ = \quad & \frac{ad}{bd}+\frac{bc}{bd} \\ = \quad & \frac{ad+bc}{bd} \end{aligned}$$

$$\begin{aligned} & 3+\frac{4\cdot 8}{5} \\ = \quad & \frac{3}{1}+\frac{4}{5} \\ = \quad & \frac{3\bullet}{1\bullet} \\ = \quad & \frac{15}{5}+\frac{4}{5} \\ = \quad & \frac{19}{5} \end{aligned}$$

$$\begin{aligned} & x+\frac{y}{z} \\ = \quad & \frac{x}{1}+\frac{y}{z} \\ = \quad & \frac{x\bullet}{1\bullet} \\ = \quad & \frac{xz}{z}+\frac{y}{z} \\ = \quad & \frac{xz+y}{z} \end{aligned}$$

To **add** or **subtract** fractions with <u>different</u> denominators, each fraction must be rewritten with the same denominator. This is accomplished by multiplying one or both fractions by a fraction equal to 1. A fraction is equal to 1 when the top and bottom are the same (and assuming the top and bottom are <u>not</u> zeros).

In the first example above, we see that we needed to change each denominator into 21. We did this by multiplying the top and bottom of the first fraction by 7, and the top and bottom of the second fraction by 3.

Similarly, in the second example, to achieve a common denominator we multiplied the top and bottom of the first fraction by d, and the top and bottom of the second fraction by b. This converted both fractions into fractions with the same denominator, bd. Then they were ready to be added together.

Homework

7. Add or subtract the fractions:

a. $\frac{1}{2}+\frac{1}{3}$ b. $\frac{2}{5}-\frac{1}{10}$ c. $\frac{1}{4}-\frac{5}{6}$ d. $\frac{1}{3}-\frac{9}{2}$

e. $\frac{a}{b}+\frac{w}{x}$ f. $\frac{c}{d}-\frac{g}{h}$ g. $\frac{m}{n}+\frac{m}{q}$ h. $\frac{a}{b}-\frac{a}{c}$

i. $\frac{6}{a}+\frac{3}{b}$ j. $\frac{a}{R}-\frac{3}{T}$ k. $a+\frac{b}{c}$ l. $w-\frac{x}{y}$

m. $\frac{k}{j}+n$ n. $\frac{w}{x}-z$ o. $\frac{x^2}{a}+\frac{y^2}{a}$ p. $\frac{w}{x}+\frac{y}{z}$

❑ *MULTIPLYING AND DIVIDING FRACTIONS*

$$\frac{7}{10}\cdot\frac{11}{15}=\frac{77}{150} \quad \Rightarrow \quad \frac{m}{n}\cdot\frac{p}{q}=\frac{mp}{nq}$$

To **multiply** fractions

i) the numerator of the answer is the product of the numerators

ii) the denominator of the answer is the product of the denominators

$$\frac{2}{7}\div\frac{5}{9}=\frac{2}{7}\times\frac{9}{5}=\frac{18}{35} \quad\Longrightarrow\quad \frac{c}{d}\div\frac{u}{w}=\frac{c}{d}\times\frac{w}{u}=\frac{cw}{du}$$

To **divide** fractions, multiply the first fraction (the *dividend*) by the reciprocal of the second fraction (the *divisor*).

Shortcut: When multiplying fractions, before actually multiplying the tops and bottoms, sometimes dividing out common factors can simplify the reducing of the final answer.

$$\frac{5}{6}\times\frac{13}{5}=\frac{\cancel{5}^{1}}{6}\times\frac{13}{\cancel{5}_{1}}=\frac{13}{6} \quad\Longrightarrow\quad \frac{x}{y}\times\frac{z}{x}=\frac{\cancel{x}^{1}}{y}\times\frac{z}{\cancel{x}_{1}}=\frac{z}{y}$$

Homework

8. Perform the indicated operation:

a. $\frac{2}{3}\cdot\frac{5}{9}$ b. $\frac{1}{2}\times\frac{5}{7}$ c. $\left(\frac{8}{9}\right)\left(\frac{9}{10}\right)$ d. $\frac{4}{5}\cdot\frac{5}{4}$

e. $\frac{w}{x}\cdot\frac{w}{z}$ f. $\frac{a}{b}\times\frac{c}{b}$ g. $\frac{x}{y}\cdot\frac{c}{d}$ h. $\left[\frac{a}{b}\right]\left[\frac{b}{a}\right]$

i. $a\times\frac{b}{c}$ j. $\frac{m}{n}\times Q$ k. $z\div\frac{w}{z}$ l. $\frac{u}{w}\div a$

m. $\frac{x}{y}\div\frac{x}{y}$ n. $\frac{a}{b}\div\frac{b}{a}$ o. $\frac{a}{b}\div\frac{a}{c}$ p. $\frac{g}{h}\cdot\frac{g}{h}$

q. $3\cdot\frac{x}{y}$ r. $a\left(\frac{b}{c}\right)$ s. $\left(\frac{w}{7}\right)A$ t. $\frac{w}{7}+A$

9. Perform the indicated operation:

a. $\frac{K}{L} \div \frac{K}{M}$ b. $\frac{n}{b} + \frac{n}{b}$ c. $\frac{x}{a} - \frac{x}{a}$ d. $\frac{a}{b} \cdot \frac{b}{c}$

e. $\frac{G}{H} \div \frac{G}{H}$ f. $\frac{1}{a} + \frac{1}{b}$ g. $\frac{2}{c} - \frac{3}{d}$ h. $\frac{a}{w} - \frac{a}{z}$

i. $\frac{R}{T} \times \frac{T}{R}$ j. $6 + \frac{x}{y}$ k. $\frac{u}{m} - n$ l. $\frac{x}{y} \div \frac{x}{z}$

m. $\left(\frac{b}{c}\right)\left(\frac{b}{c}\right)$ n. $\frac{b}{c} \times \frac{c}{b}$ o. $\frac{m}{n} \div \frac{p}{q}$ p. $\frac{p}{q} + \frac{m}{n}$

q. $a\left(\frac{b}{c}\right)$ r. $\frac{w}{n} \div 5$ s. $\frac{p}{m} + E$ t. $\frac{x}{y} + \frac{a}{b} \cdot \frac{b}{a}$

Review Problems

10. a. What is the reciprocal of $-\frac{2}{9}$? b. What is the reciprocal of 0?

11. Perform the indicated operation:

a. $\frac{a}{b} + \frac{c}{b}$ b. $\frac{x}{y} + \frac{y}{x}$ c. $a - \frac{w}{u}$ d. $\frac{m}{n} - A$

e. $\frac{x}{y} \cdot \frac{w}{z}$ f. $\frac{x}{z} \times \frac{z}{y}$ g. $\frac{a}{b} \div \frac{c}{d}$ h. $c \div \frac{d}{e}$

i. $\frac{g}{h} \div \pi$ j. $\frac{m}{\pi} \cdot \frac{\pi}{m}$ k. $h + \frac{k}{h}$ l. $\frac{a}{b}\left(\frac{b}{c}\right)$

m. $\left(\frac{a}{t}\right)\left(\frac{t}{a}\right)$ n. $\frac{a}{n^2} + \frac{b}{n^2}$ o. $\frac{8}{ab} + \frac{6}{ab}$ p. $\frac{a}{x^3} - \frac{b}{x^3}$

q. $\frac{w}{4c}+\frac{w}{4c}$ r. $\frac{a}{mn}-\frac{a}{mn}$ s. $\frac{A}{B}\cdot\frac{C}{D}$ t. $\frac{C}{D}\div\frac{B}{A}$

u. $\frac{s}{r}\div s$ v. $Q\div\frac{Q}{R}$ w. $\frac{6}{L}\div\frac{6}{L}$ x. $\frac{M}{Q}\div\frac{Q}{M}$

y. $\frac{ab}{xy}+\frac{c}{xy}$ z. $\frac{abc}{d}-\frac{def}{d}$

Solutions

1. a. 5 b. 11 c. 18 d. 2 e. 0
f. Undefined

2. a. 0.125 b. 0.1875 c. 0.03125 d. 4.3 e. 1.8
f. Undefined

3. The second one

4. a. $\frac{2}{3}$ b. $-\frac{5}{4}$ c. $-\frac{7}{8}$ d. $\frac{1}{7}$ e. $-\frac{1}{14}$
f. -7 g. $\frac{1}{a}$ h. x i. $\frac{y}{x}$ j. $-\frac{x}{y}$
k. $-\frac{1}{T}$ l. $-w$ m. 1 n. -1 o. Undefined

5. a. $-\frac{19}{7}$ b. $-\frac{7}{19}$ c. $\left(-\frac{17}{9}\right)\left(-\frac{9}{17}\right)=\left(-\frac{{}^{1}\cancel{17}}{\cancel{9}_{1}}\right)\left(-\frac{\cancel{9}^{1}}{\cancel{17}_{1}}\right)=1$ ✓

6. a. $\frac{9}{10}$ b. $\frac{87}{101}$ c. $\frac{1}{2}$ d. $-\frac{2}{5}$ e. $\frac{x+z}{y}$
f. $\frac{m-q}{u}$ g. $\frac{c+d}{a}$ h. $\frac{t-z}{w}$ i. $\frac{b+7}{c}$ j. $\frac{2}{Q}$
k. $-\frac{5}{R}$ l. $\frac{8}{3a}$ m. $\frac{a-d}{bc}$ n. $\frac{a+b}{x^2}$ o. $\frac{7}{10x}$
p. $\frac{u-b}{a^3}$

7. a. $\frac{5}{6}$ b. $\frac{3}{10}$ c. $-\frac{7}{12}$ d. $-\frac{25}{6}$

e. $\frac{ax+bw}{bx}$ f. $\frac{ch-dg}{dh}$ g. $\frac{mq+mn}{nq}$ h. $\frac{ac-ab}{bc}$

i. $\frac{6b+3a}{ab}$ j. $\frac{aT-3R}{RT}$ k. $\frac{ac+b}{c}$ l. $\frac{wy-x}{y}$

m. $\frac{k+jn}{j}$ n. $\frac{w-xz}{x}$ o. $\frac{x^2+y^2}{a}$ p. $\frac{wz+xy}{xz}$

8. a. $\frac{10}{27}$ b. $\frac{5}{14}$ c. $\frac{4}{5}$ d. 1

e. $\frac{w^2}{xz}$ f. $\frac{ac}{b^2}$ g. $\frac{cx}{dy}$ h. 1

i. $\frac{ab}{c}$ j. $\frac{mQ}{n}$ k. $\frac{z^2}{w}$ l. $\frac{u}{aw}$

m. 1 n. $\frac{a^2}{b^2}$ o. $\frac{c}{b}$ p. $\frac{g^2}{h^2}$

q. $\frac{3x}{y}$ r. $\frac{ab}{c}$ s. $\frac{Aw}{7}$ t. $\frac{w+7A}{7}$

9. a. $\frac{M}{L}$ b. $\frac{2n}{b}$ c. 0 d. $\frac{a}{c}$

e. 1 f. $\frac{a+b}{ab}$ g. $\frac{2d-3c}{cd}$ h. $\frac{az-aw}{wz}$

i. 1 j. $\frac{x+6y}{y}$ k. $\frac{u-mn}{m}$ l. $\frac{z}{y}$

m. $\frac{b^2}{c^2}$ n. 1 o. $\frac{mq}{np}$ p. $\frac{np+mq}{qn}$

q. $\frac{ab}{c}$ r. $\frac{w}{5n}$ s. $\frac{p+Em}{m}$ t. $\frac{x+y}{y}$

10. a. $-\frac{9}{2}$ b. 0 does not have a reciprocal.

11. a. $\frac{a+c}{b}$ b. $\frac{x^2+y^2}{xy}$ c. $\frac{au-w}{u}$ d. $\frac{m-An}{n}$

e. $\frac{wx}{yz}$ f. $\frac{x}{y}$ g. $\frac{ad}{bc}$ h. $\frac{ce}{d}$

i. $\frac{g}{\pi h}$ j. 1 k. $\frac{h^2+k}{h}$ l. $\frac{a}{c}$

m. 1 n. $\frac{a+b}{n^2}$ o. $\frac{14}{ab}$ p. $\frac{a-b}{x^3}$

q. $\frac{w}{2c}$ r. 0 s. $\frac{AC}{BD}$ t. $\frac{AC}{BD}$

u. $\frac{1}{r}$ v. R w. 1 x. $\frac{M^2}{Q^2}$

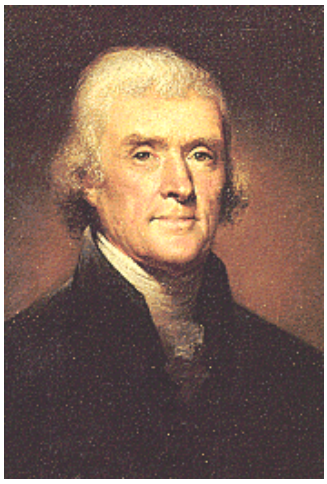
y. $\frac{ab+c}{xy}$ z. $\frac{abc-def}{d}$

❑ *To ∞ and Beyond*

A. $\frac{1}{a}+\frac{2}{b}+\frac{3}{c}=$

B. $\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right)=$

"On matters of style, swim with the current; on matters of principle, stand like a rock."



Thomas Jefferson