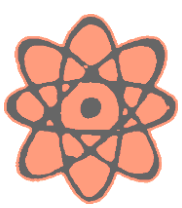
# THE GROWTH AND DECAY FORMULA

## ❑ INTRODUCTION

Many things grow and shrink (decay) at a rate that's based on how much of the stuff there is at the moment.

For instance, the number of births in a population is based on the number of people living at the time (since the more people there are, the more new people there will be).

Another example is compound interest: As money accumulates (due to earned interest) in the account, the more interest the account will earn; that is, the interest will earn interest.



In science, we can observe the radioactive decay of an element like uranium. As the uranium begins to disintegrate, less of it will decay, because as time goes on, there's less of it left to decay.

## ❑ THE GROWTH AND DECAY FORMULA

Here's a formula to help you predict how much of something there will be in the future. It works for population growth, continuous compounding of interest, and the decay of radioactive substances:

Let

$A_0$ = starting amount (read: "$A$ sub zero" or "$A$ naught")
$A$ = ending amount
$e$ = a constant whose value is **approximately** 2.718.
$k$ = growth or decay rate (expressed as a decimal)
$t$ = time

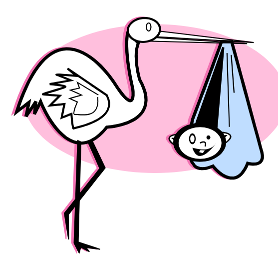
Then

$$A = A_0 e^{kt}$$

Get your calculator out. Here's a quick review of the "exponent" button. To calculate $(3.2)^{4.15}$, try either

3.2 [$y^x$] 4.15 = OR 3.2 [^] 4.15 =

The answer is about 124.845.



EXAMPLE 1: **Assuming an initial population of 1,506, and a growth rate of 25% per year compounded continuously, predict the population in 10 years.**

Solution: Let's start by writing the formula that will solve this problem:

$$A = A_0 e^{kt}$$

The initial population is 1,506; so $A_0 = 1{,}506$.

The growth rate is 25%; thus $k = 0.25$.

We're talking about a period of 10 years; therefore $t = 10$.

Plug all these values into our formula (using 2.718 for $e$):

$$A = 1{,}506(2.718)^{(0.25)(10)}$$

$$\Rightarrow \quad A = 1{,}506(2.718)^{2.5}$$

$$\Rightarrow \quad A = 1{,}506(12.18) \qquad \text{(we'll round this result to 2 digits)}$$

$$\Rightarrow \quad \boxed{A = 18{,}343}$$

<u>EXAMPLE 2:</u> **Assuming a starting investment of \$2,401, and an annual interest rate of 13% compounded continuously, predict the value of the investment in 11 years.**

<u>Solution:</u> The formula we used for population growth works equally well for compound interest.

$$A = A_0 e^{kt}$$

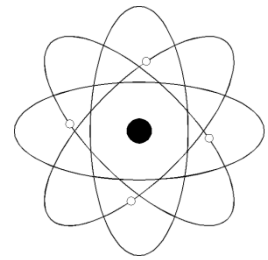
$$\Rightarrow \quad A = 2{,}401(2.718)^{(0.13)(11)}$$

$$\Rightarrow \quad A = 2{,}401(2.718)^{1.43}$$

$$\Rightarrow \quad A = 2{,}401(4.18) \qquad \text{(we'll round this result to 2 digits)}$$

$$\Rightarrow \quad \boxed{A = \$10{,}036}$$

<u>EXAMPLE 3:</u> **Starting with 624 grams of uranium, and assuming an annual <u>decay</u> rate of 5%, compute the number of grams of uranium remaining after 9 years.**



<u>Solution:</u> We use the same growth/decay formula except for one thing: Since the amount of uranium is shrinking (decaying) rather than growing, we will use a decay rate of **–5%** (that's <u>negative</u> 5 percent) in our formula:

$$A = A_0e^{kt}$$

$$\Rightarrow \quad A = 624(2.718)^{(-0.05)(9)}$$

$$\Rightarrow \quad A = 624(2.718)^{-0.45}$$

$$\Rightarrow \quad A = 624(0.64)$$ (we'll round this result to 2 digits)

$$\Rightarrow \quad \boxed{A = 399 \text{ grams}}$$

The number ***e*** used in the growth formula is one of the most important numbers in math, science, engineering, and business. Like the number $\pi$, this real number contains an infinite number of digits which never form a repeating pattern (called *irrational*).

# Homework

Note: The result of raising 2.718 to the exponent is assumed to be rounded to 2 digits, as in the examples.

1. Starting with 178 grams of uranium, and assuming an annual decay rate of 17%, predict the number of grams remaining after 18 years.
2. Assuming an initial population of 3,902, and a growth rate of 14% per year, determine the population in 10 years.
3. Assuming an initial investment of $1,299, and an annual interest rate of 14% compounded continuously, predict the value of the investment in 4 years.
4. Assuming an initial investment of $9,574, and an annual interest rate of 11% compounded continuously, compute the value of the investment in 2 years.
5. Starting with 416 grams of thorium, and assuming an annual decay rate of 3%, determine the number of grams remaining after 26 years.
6. Assuming an initial population of 2,586, and a growth rate of 16% per year, predict the population in 4 years.
7. Assuming an initial population of 1,422, and a growth rate of 12% per year, compute the population in 2 years.
8. Assuming an initial investment of $6,897, and an annual interest rate of 14% compounded continuously, compute the value of the investment in 6 years.
9. Starting with 215 grams of plutonium, and assuming an annual decay rate of 15%, calculate the number of grams remaining after 26 years.
10. Assuming an initial population of 2,159, and a growth rate of 6% per year, calculate the population in 11 years.

[ Technically speaking, the growth/decay formula we've been using applies only when the growth or decay is *continuous*, which means the growth or decay occurs at every single moment of time. This may not strictly be the case in all situations (for example, in the births of people), but let's not worry about it in this chapter. Let's just use the formula to solve the problems. ]

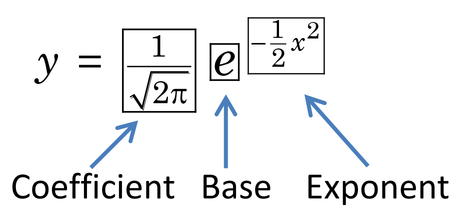
## ❑ *THE BELL-SHAPED CURVE (ADVANCED)*

Some of you will be taking Statistics, and you will learn that the most important function in the course is a special bell-shaped curve called the ***standard normal*** curve. It's an exponential function, and its formula is quite complicated, but we have all the tools needed to find some of its points and then sketch it.

The formula for the standard normal curve is given by

$$y = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$$

Note that this function is called an exponential function because the base is a constant (in this case, $e$) and the variable $x$ occurs in the exponent. Let's decipher this formula before we try to plot any points. The coefficient of the exponential function is $\frac{1}{\sqrt{2\pi}}$, the base is $e$, and the exponent on the $e$ is $-\frac{1}{2}x^2$.




Let's start calculating some $(x, y)$ pairs for this function and see what kind of graph we get. Let's start by letting $x = 0$. This, of course, will yield the $y$-intercept.

$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0)^2} = \frac{1}{\sqrt{2\pi}}e^0 = \frac{1}{\sqrt{2\pi}}(1) \approx 0.3989$$

This gives us the approximate point **(0, 0.4)**. We're off to a good start.

Now we'll choose $x = 1$. Plugging this number into the function gives

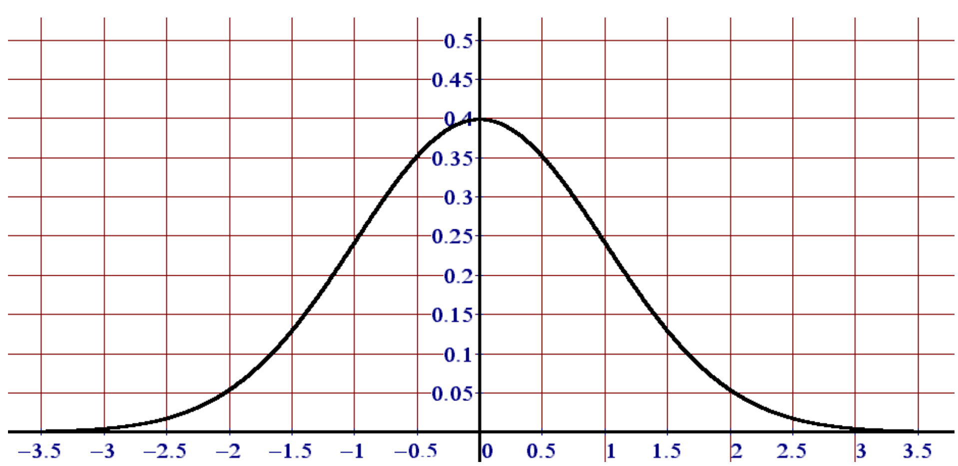
$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(1)^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}(0.6065) \approx 0.2420$$

Our second point is therefore **(1, 0.24)**. Now for $x = -1$. You should be able to do this one in your head. First the $-1$ is squared, giving 1 -- but this is exactly what we had in the previous calculation, and so the $y$-value must also be 0.2420. Our third point is therefore **(–1, 0.24)**.

Choosing $x = 2$ (or $-2$) gives the following calculation for $y$:

$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\pm 2)^2} = \frac{1}{\sqrt{2\pi}}e^{-2} = \frac{1}{\sqrt{2\pi}}(0.1353) \approx 0.0540$$

We now have two more points for our graph: **(2, 0.05)** and **(–2, 0.05)**. Continuing in this manner we can find the points **(3, 0.0044)** and **(–3, 0.0044)**. It appears that as we get farther from the origin, the $y$-values are getting smaller and smaller, but always remaining positive. If we plot the seven points we've just computed and then connect them with a smooth curve, we get the following:



# Solutions

| | | | |
|---|---|---|---|
| **1**. 9 g | **2**. 15,803 | **3**. $2,273 | **4**. $11,968 |
| **5**. 191 g | **6**. 4,913 | **7**. 1,806 | **8**. $16,001 |
| **9**. 4 g | **10**. 4,167 | | |

"Education is not the
filling of a pail,
but the lighting of a fire."

– William Butler Yeats