PARALLEL AND PERPENDICULAR LINES

❑ INTRODUCTION

It's time we revisit our friend, the equation of a line:

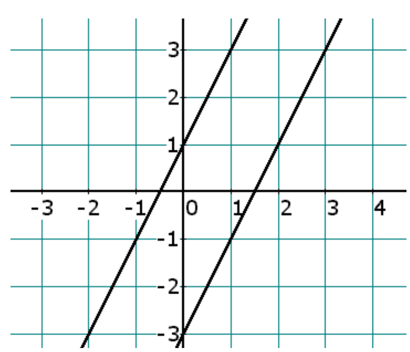
$$y = \underbrace{m}_{\text{SLOPE}}x + \underbrace{b}_{y\text{-INTERCEPT}}$$

To be precise, *b* is not the *y*-intercept; *b* is the *y*-coordinate of the *y*-intercept. The *y*-intercept is properly written (0, *b*).

❑ PARALLEL LINES

Let's begin by assuming that throughout this chapter we will never be referring to horizontal or vertical lines -- these kinds of lines are covered in detail in the chapter Horizontal and Vertical Lines.

Let's graph the two lines $y = 2x + 1$ and $y = 2x - 3$ on the same grid:



Notice that the two lines appear to be parallel (and they really are). Now, what do the equations of the two lines have in common? The formulas for the lines show that each line has a slope of 2. Since slope is a measure of steepness, does it seem reasonable that if two lines are parallel, then they are equally steep, and therefore they must have the same slope?

Parallel lines have the **same** slope.

For a simple example: Suppose Line 1 has a slope of –9, and also suppose that Line 2 is parallel to Line 1. We can then deduce (conclude by logic) that the slope of Line 2 is also –9, without even graphing it.

EXAMPLE 1: **Find the slope of any line which is *parallel* to the line $7x - 5y = 2$.**

Solution: Any line which is parallel to the line $7x - 5y = 2$ must have the *same* slope as the line $7x - 5y = 2$. So, if we can compute the slope of this line, we will have the slope of any line parallel to it. The easiest way to find the slope of the line is to convert it to $y = mx + b$ form:

$$7x - 5y = 2 \Rightarrow -5y = -7x + 2 \Rightarrow y = \frac{7}{5}x - \frac{2}{5}$$

The slope of the given line is $\frac{7}{5}$, and so we conclude that any line which is parallel to the line $7x - 5y = 2$ must have a slope of

$$\boxed{\frac{7}{5}}$$

NOTE: The examples which follow are solved using the $y = mx + b$ form of a line. However, if you've learned the point-slope form of a line, $y - y_1 = m(x - x_1)$, then you might want to use that form instead.

EXAMPLE 2: **Find the equation of the line which is *parallel* to the line $3x + y = 5$, and which passes through the point (6, 2).**

Solution: We're looking for an unknown line

$$y = mx + b$$

The slope of our unknown line was not given to us, but we know that it's the same as the slope of the given line, since the two lines are parallel. Solving the given line for y yields the line $y = -3x + 5$, whose slope is clearly -3. So the slope of our unknown line is also -3. At this point in the problem we can write our line as

$$y = -3x + b$$

Plugging the given point (6, 2) into this equation allows us to find b:

$$\mathbf{2} = -3(\mathbf{6}) + b$$

$$\Rightarrow \quad 2 = -18 + b$$

$$\Rightarrow \quad 20 = b$$

and we're done; our line is

$$\boxed{y = -3x + 20}$$

Parallel lines have so much in common – it's a shame they'll never meet.

Homework

1. a. A given line has a slope of 7. What is the slope of any line that is parallel to the given line?

   b. A given line has a slope of $-\frac{2}{3}$. What is the slope of any line that is parallel to the given line?

2. a. What is the slope of any line that is parallel to the line $y = \frac{3}{4}x - 9$?

   b. What is the slope of any line that is parallel to the line $5x + 2y = 9$?

3. a. Prove that the lines $2x - 4y = 5$ and $3x - 6y = 1$ <u>are</u> parallel.

   b. Prove that the lines $3x - y = 4$ and $5x + 2y = 10$ are <u>not</u> parallel.

4. Find the equation of the line which is *parallel* to the given line, and which passes through the given point:

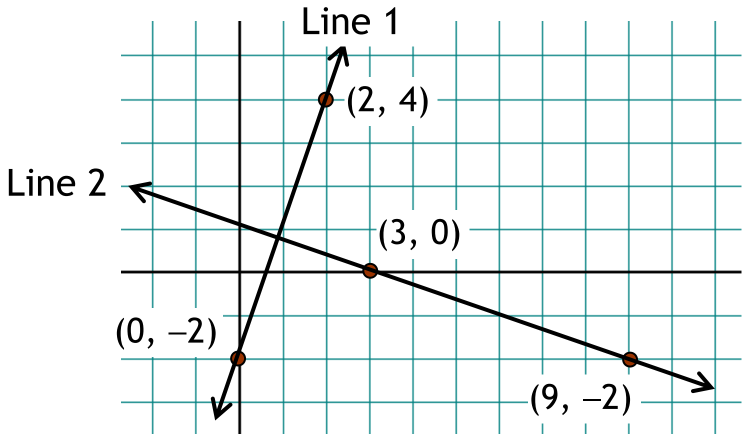
   a. $y = 3x + 4$; $(3, -5)$ b. $y = \frac{1}{2}x - 5$; $(-1, 7)$

   c. $3x + y = 7$; $(1, 9)$ d. $2x - y = 8$; $(-5, -4)$

   e. $2x - 3y = 1$; $(2, -5)$ f. $3x + 4y = 10$; $(8, 0)$

❑ *PERPENDICULAR LINES*

Parallel lines have the same slope -- certainly *perpendicular* lines do not! But is there some relationship, some connection, between the slopes of perpendicular lines? Let's see if we can discover one with an example. In the following grid, Line 1 is perpendicular to Line 2, and points have been labeled so that we can easily calculate m_1 and m_2, the slopes of the two lines.



First we compute the slope of Line 1:

$$m_1 = \frac{\Delta y}{\Delta x} = \frac{4-(-2)}{2-0} = \frac{6}{2} = \mathbf{3}$$

Next we compute the slope of Line 2:

$$m_2 = \frac{\Delta y}{\Delta x} = \frac{0-(-2)}{3-9} = \frac{2}{-6} = -\frac{\mathbf{1}}{\mathbf{3}}$$

There are two things to note regarding these two slopes of the two perpendicular lines. First, one slope is positive while the other is negative. This makes sense because as we move from left to right, Line 1 is increasing while Line 2 is decreasing.

Second, the slope of Line 1, m_1, is kind of a big number (the line's pretty steep), while the slope of Line 2, m_2, (ignoring the minus sign) is a relatively small number (the line's not very steep).

Specifically, the two slopes have opposite signs, and they are also (ignoring the minus sign) *reciprocals* of each other. In other words, when looking at the slopes of two **perpendicular lines**, each of the slopes is the ***opposite reciprocal*** of the other.

Although we have shown this relationship (*opposite reciprocal*) for just this example, it can be proved that this relationship always works for perpendicular lines.

Perpendicular lines have slopes that are **opposite reciprocals** of each other.

For example, if a line has a slope of $\frac{7}{4}$, then any perpendicular line must have a slope of $-\frac{4}{7}$.

And consider the line $y = -5x + 1$. Since its slope is -5, it follows that the slope of any perpendicular line must be $\frac{1}{5}$.

Some books say that the slopes of two perpendicular lines are *negative reciprocals* of each other.

Alternative: We've learned that the slopes of two perpendicular lines are *opposite reciprocals* of each other. But some books say that two lines are perpendicular if their slopes have a product of –1. Do both of these rules mean the same thing? Yes -- assume that the product of their slopes is –1:

$$m_1 m_2 = -1$$

Solving for m_1 gives us the equation

$$m_1 = -\frac{1}{m_2},$$

which says that one slope is the *opposite reciprocal* of the other.

Homework

5. A given line has a slope of $-\frac{5}{3}$. What is the slope of any line that is perpendicular to the given line?

6. Prove that the lines $7x - 2y = 10$ and $4x + 14y = 23$ are perpendicular.

7. Prove that the lines $3x + 2y = 10$ and $2x + 3y = 9$ are not perpendicular.

8. Prove that the lines $5x - 3y = 10$ and $5x + 3y = 17$ are not perpendicular.

9. Find the slope of any line which is *perpendicular* to the given line:

   a. $y = -7x + 9$ b. $y = \frac{5}{4}x + 10$

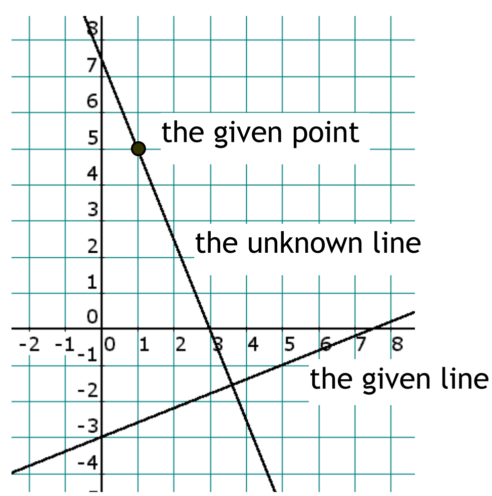
   c. $2x + 7y = 10$ d. $3x - 2y = 0$

EXAMPLE 3: **Find the equation of the line which is *perpendicular* to the line $2x - 5y = 15$, and which passes through the point (1, 5).**

Solution: We're looking for an unknown line

$$y = mx + b$$

The slope of our unknown line was not given to us, but we know that it's the *opposite reciprocal* of the slope of the given line, since the two lines are perpendicular. To determine the slope of the given line, we solve for y:



$$2x - 5y = 15$$

$$\Rightarrow \quad -5y = -2x + 15$$

$$\Rightarrow \quad \frac{-5y}{-5} = \frac{-2x+15}{-5}$$

$$\Rightarrow \quad y = \frac{2}{5}x - 3$$

telling us that the slope of the given line is $\frac{2}{5}$. So the slope of our unknown line is the opposite reciprocal of that, which is $-\frac{5}{2}$. At this point in the problem we can write our line as

$$y = -\frac{5}{2}x + b$$

Plugging the given point (1, 5) into this equation allows us to find b:

$$\mathbf{5} = -\frac{5}{2}(\mathbf{1}) + b \quad \Rightarrow \quad 5 + \frac{5}{2} = b \quad \Rightarrow \quad b = \frac{15}{2}$$

and we're done; our line is

$$\boxed{y = -\frac{5}{2}x + \frac{15}{2}}$$

Homework

10. Find the equation of the line which is *perpendicular* to the given line, and which passes through the given point:

a. $y = 3x + 4$; $(3, -5)$

b. $y = \frac{1}{2}x - 5$; $(-1, 7)$

c. $3x + y = 7$; $(1, 9)$

d. $2x - y = 8$; $(-5, -4)$

e. $5x - 3y = 1$; $(2, -5)$

f. $3x + 4y = 10$; $(8, 0)$

Review Problems

11. The slopes of two parallel lines are ______.

12. The slopes of two perpendicular lines are ______.

13. The slope of a line is $\frac{5}{7}$. What is the slope of any parallel line?

14. The slope of a line is $-\frac{4}{9}$. What is the slope of any perpendicular line?

15. T/F: The lines $y = 7x - 3$ and $14x - 2y = 22$ are parallel.

16. T/F: The lines $3x - 7y = 1$ and $7x + 3y = 0$ are perpendicular.

17. Find the equation of the line which is parallel to $3x - 7y = 9$ and passes through the point $(-3, 10)$.

18. Find the equation of the line which is perpendicular to $4x - 9y = 11$ and passes through the point $(3, -13)$.

19. Find the equation of the line which is parallel to the line $3x - 4y = 1$ and which passes through the point $(-2, -7)$.

20. Find the equation of the line which is perpendicular to the line $3x - 4y = 1$ and which passes through the point $(-2, -7)$.

21. Which one of the following lines is *parallel* to the line $5x - 3y = 7$?

    a. $y = \frac{3}{5}x - 1$ b. $y = -\frac{3}{5}x + 4$ c. $y = \frac{5}{3}x - 3$

    d. $y = -\frac{5}{3}x + 2$ e. $y = \frac{5}{3}$

22. Which one of the following lines is *perpendicular* to the line $5x - 3y = 7$?

    a. $y = \frac{3}{5}x - 1$ b. $y = -\frac{3}{5}x + 4$ c. $y = \frac{5}{3}x - 3$

    d. $y = -\frac{5}{3}x + 2$ e. $y = \frac{5}{3}$

Solutions

1. a. 7 b. $-\frac{2}{3}$

2. a. $\frac{3}{4}$ b. $-\frac{5}{2}$

3. a. Each line has a slope of $\frac{1}{2}$. Same slope $\Rightarrow$ parallel lines.

b. The slopes are 3 and –5/2. Different slopes $\Rightarrow$ non-parallel lines.

4. a. $y = 3x - 14$ b. $y = \frac{1}{2}x + \frac{15}{2}$ c. $y = -3x + 12$

d. $y = 2x + 6$ e. $y = \frac{2}{3}x - \frac{19}{3}$ f. $y = -\frac{3}{4}x + 6$

5. $\frac{3}{5}$

6. The slopes are $\frac{7}{2}$ and $-\frac{2}{7}$, which are opposite reciprocals of each other.

7. The slopes are $-\frac{3}{2}$ and $-\frac{2}{3}$, which are <u>not</u> opposite reciprocals of each other. (They're reciprocals, but not opposites.)

8. The slopes are $\frac{5}{3}$ and $-\frac{5}{3}$, which are <u>not</u> opposite reciprocals of each other. (They're opposites, but not reciprocals.)

9. a. $\frac{1}{7}$ b. $-\frac{4}{5}$ c. $\frac{7}{2}$ d. $-\frac{2}{3}$

10. a. $y = -\frac{1}{3}x - 4$ b. $y = -2x + 5$ c. $y = \frac{1}{3}x + \frac{26}{3}$

d. $y = -\frac{1}{2}x - \frac{13}{2}$ e. $y = -\frac{3}{5}x - \frac{19}{5}$ f. $y = \frac{4}{3}x - \frac{32}{3}$

11. equal **12.** opposite reciprocals

13. $\frac{5}{7}$

14. $\frac{9}{4}$

15. T

16. T

17. $y = \frac{3}{7}x + \frac{79}{7}$

18. $y = -\frac{9}{4}x - \frac{25}{4}$

19. $y = \frac{3}{4}x - \frac{11}{2}$

20. $y = -\frac{4}{3}x - \frac{29}{3}$

21. c.

22. b.

"Upon the subject of education ... I can only say that I view it as the most important subject which we as a people may be engaged in."

– *Abraham Lincoln*