DIVIDING POLYNOMIALS

❑ INTRODUCTION

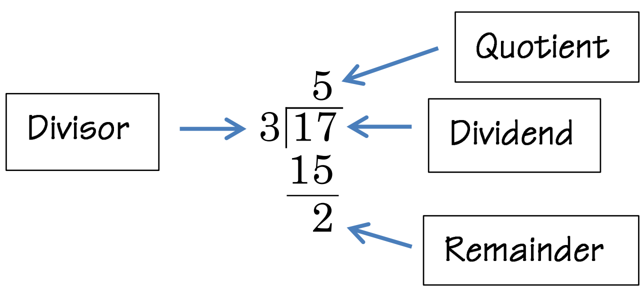
First we need the right terminology. When written as a fraction, a division problem has two parts:

$$\frac{\text{dividend}}{\text{divisor}}$$

When written in the standard "long division" format, we write

$$\text{divisor}\overline{\big)\text{dividend}}$$

The result of dividing is called the ***quotient***, and the leftover is called the ***remainder***. For example,



We can then write the answer as $5+\frac{2}{3}$ $\left(\text{quotient} + \frac{\text{remainder}}{\text{divisor}}\right)$, which is written as the mixed number $5\frac{2}{3}$ when we're dealing with numbers.

❑ DIVIDING A POLYNOMIAL BY A MONOMIAL

Just as $\frac{1}{7}+\frac{3}{7} = \frac{4}{7}$, we can work the problem $\frac{a}{b}+\frac{c}{b}$ by adding the numerators, and placing that sum over the common denominator b:

$$\frac{a}{b}+\frac{c}{b} = \frac{a+c}{b}$$

By reversing this reasoning, we can take the fraction $\frac{a+c}{b}$ and, if we like, *split* it into the sum of two fractions:

$$\frac{a+c}{b} = \frac{a}{b}+\frac{c}{b}$$

This is the trick we need to divide a polynomial by a monomial.

EXAMPLE 1: **Divide:** $\frac{\mathbf{12x^2y^3 - 8x^3y^2 + 7xy^3}}{\mathbf{2x^2y}}$

Solution: Split the fraction into three separate fractions:

$$\frac{12x^2y^3}{2x^2y} - \frac{8x^3y^2}{2x^2y} + \frac{7xy^3}{2x^2y}$$

and then simplify (reduce) each fraction:

$$\boxed{6y^2 - 4xy + \frac{7y^2}{2x}}$$

Homework

1. Perform each division problem, where the divisor (the bottom) is a monomial:

a. $\frac{x^3 - x^2 + x}{x}$ b. $\frac{14xy + 21x^2y - 28xy^2}{7xy}$ c. $\frac{x^2 + 3x + 1}{x}$

d. $\frac{a+b}{b}$ e. $\frac{x-y}{y}$ f. $\frac{ax+bx}{x}$

❑ *DIVIDING A POLYNOMIAL BY A POLYNOMIAL*

Think back when you were a kid and learned long division of numbers. Though I've seen different ways of doing this, the standard method boils down to a 4-step process, a process that is repeated until the problem is finished:

1. Divide the divisor into the first part of the dividend
2. Multiply the part of the quotient calculated in step 1 by the divisor
3. Subtract
4. Bring down the next digit

And then repeat steps 1 – 4 as many times as necessary.

We use the same process for polynomial long division in algebra.

EXAMPLE 2: **Perform the long division:** $\dfrac{\mathbf{3x^3-5x-2}}{\mathbf{x-1}}$

Solution: The first step is to fill in the missing term in the dividend. Since there is no x^2 term, we put in the "place-holder" $0x^2$ between the cubic term and the linear term, giving us a dividend of $3x^3 + 0x^2 - 5x - 2$. So our long division problem is

$$x-1\overline{\smash{)}\,3x^3+0x^2-5x-2}$$

1. Divide x into $3x^3$; it goes in $3x^2$ times (since $3x^2 \cdot x = 3x^3$):

$$\begin{array}{r} \mathbf{3x^2} \\ \mathbf{x}-1\overline{\smash{)}\,\mathbf{3x^3}+0x^2-5x-2} \end{array}$$

2. <u>Multiply</u> $3x^2$ by the divisor, $x - 1$:

$$
\begin{array}{r|l}
 & \mathbf{3x^2} \\
\cline{2-2}
\mathbf{x-1} & 3x^3 + 0x^2 - 5x - 2 \\
 & \underline{\mathbf{3x^3 - 3x^2}}
\end{array}
$$

3. <u>Subtract</u>; $3x^3 - 3x^3 = 0$; $0x^2 - (-3x^2) = 3x^2$:

$$
\begin{array}{r|l}
 & 3x^2 \\
\cline{2-2}
x-1 & \mathbf{3x^3} + \mathbf{0x^2} - 5x - 2 \\
 & \underline{-\,(\mathbf{3x^3 - 3x^2})} \\
 & \quad \mathbf{0} \;\; + \mathbf{3x^2}
\end{array}
$$

4. <u>Bring down</u> the next term, $-5x$:

$$
\begin{array}{r|l}
 & 3x^2 \\
\cline{2-2}
x-1 & 3x^3 + 0x^2 - \mathbf{5x} - 2 \\
 & \underline{-\,(3x^3 - 3x^2)} \\
 & \quad 0 \;\; + 3x^2 - \mathbf{5x}
\end{array}
$$

1. And repeat: <u>Divide</u> x into $3x^2$:

$$
\begin{array}{r|l}
 & 3x^2 \;\; + \mathbf{3x} \\
\cline{2-2}
\mathbf{x}-1 & 3x^3 + 0x^2 - 5x - 2 \\
 & \underline{-\,(3x^3 - 3x^2)} \\
 & \quad 0 \;\; + \mathbf{3x^2} - 5x
\end{array}
$$

2. <u>Multiply</u> $3x$ by $x - 1$, the divisor:

$$
\begin{array}{r|l}
 & 3x^2 \;\; + \mathbf{3x} \\
\cline{2-2}
\mathbf{x-1} & 3x^3 + 0x^2 - 5x - 2 \\
 & \underline{-\,(3x^3 - 3x^2)} \\
 & \quad 0 \;\; + 3x^2 - 5x \\
 & \qquad\;\; \mathbf{3x^2 - 3x}
\end{array}
$$

3. Subtract; $3x^2 - 3x^2 = 0; -5x - (-3x) = -2x$:

$$
\begin{array}{r l}
 & 3x^2 \quad +3x \\
x-1 & \overline{)\,3x^3 + 0x^2 - 5x - 2} \\
 & \underline{-(3x^3 - 3x^2)} \\
 & \quad 0 \; + \mathbf{3x^2} - \mathbf{5x} \\
 & \qquad \underline{-(\mathbf{3x^2} - \mathbf{3x})} \\
 & \qquad\quad \mathbf{0} \;\; -\mathbf{2x}
\end{array}
$$

4. Bring down the next (and last) term, -2:

$$
\begin{array}{r l}
 & 3x^2 \quad +3x \\
x-1 & \overline{)\,3x^3 + 0x^2 - 5x - \mathbf{2}} \\
 & \underline{-(3x^3 - 3x^2)} \\
 & \quad 0 \; + 3x^2 - 5x \\
 & \qquad \underline{-(3x^2 - 3x)} \\
 & \qquad\quad 0 \;\; -2x - \mathbf{2}
\end{array}
$$

1. Divide x into $-2x$:

$$
\begin{array}{r l}
 & 3x^2 \quad +3x - \mathbf{2} \\
\mathbf{x}-1 & \overline{)\,3x^3 + 0x^2 - 5x - 2} \\
 & \underline{-(3x^3 - 3x^2)} \\
 & \quad 0 \; + 3x^2 - 5x \\
 & \qquad \underline{-(3x^2 - 3x)} \\
 & \qquad\quad 0 \;\; -\mathbf{2x} - 2
\end{array}
$$

2. Multiply –2 by $x - 1$:

$$\begin{array}{r|l} & 3x^2 + 3x - \mathbf{2} \\ \hline \boldsymbol{x} - \mathbf{1} & 3x^3 + 0x^2 - 5x - 2 \\ & -(3x^3 - 3x^2) \\ \hline & 0 + 3x^2 - 5x \\ & \quad -(3x^2 - 3x) \\ \hline & \qquad 0 - 2x - 2 \\ & \qquad\quad -\mathbf{2}\boldsymbol{x} + \mathbf{2} \end{array}$$

3. Subtract; $-2x - (-2x) = 0$; $-2 - (+2) = -4$

$$\begin{array}{r|l} & 3x^2 + 3x - 2 \\ \hline x - 1 & 3x^3 + 0x^2 - 5x - 2 \\ & -(3x^3 - 3x^2) \\ \hline & 0 + 3x^2 - 5x \\ & \quad -(3x^2 - 3x) \\ \hline & \qquad 0 - \mathbf{2}\boldsymbol{x} - \mathbf{2} \\ & \qquad -(-\mathbf{2}\boldsymbol{x} + \mathbf{2}) \\ \hline & \qquad\quad \mathbf{0} \; - \mathbf{4} \end{array}$$

There are no terms left to bring down in the dividend, so we write the remainder (the –4) over the divisor and add it to the quotient. The final answer to the long division problem is

$$\boxed{3x^2 + 3x - 2 + \frac{-4}{x - 1}}$$

Homework

2. Perform each polynomial long division problem, expressing any remainder as a fraction added to the quotient:

a. $\frac{x^2+5x+6}{x+3}$ b. $\frac{x^2-9}{x-3}$ c. $\frac{x^2+2x+1}{x+1}$

d. $\frac{n^2+n-4}{n+5}$ e. $\frac{2a^2-5a+2}{a+3}$ f. $\frac{3w^2+10}{w+5}$

g. $\frac{6b^2+b-15}{2b+3}$ h. $\frac{3y^2-9}{y+5}$ i. $\frac{10x^2+3x-7}{2x-1}$

j. $\frac{x^3+1}{x+1}$ Hint: $x^3 + 1 = x^3 + 0x^2 + 0x + 1$

k. $\frac{n^3-8}{n-2}$ l. $\frac{a^3+27}{a^2-3a+9}$

3. Perform each polynomial long division problem (Hint: there is no remainder):

a. $\frac{40x^3+97x^2+60x+27}{5x+9}$ b. $\frac{8w^3+22w^2+13w+2}{2w^2+5w+2}$

c. $\frac{40r^3-4r^2-7r-3}{8r^2+4r+1}$ d. $\frac{63m^3+43m^2+13m+1}{7m^2+4m+1}$

Practice Problems

4. Divide: $\frac{4x^3-8x^2+6x-10}{4x^2}$

5. Divide: $\frac{x^2+9}{x-5}$

6. Divide: $\frac{x^3-3x+8}{x+3}$

7. Divide: $\frac{x^4 - 1}{x + 1}$

8. Divide: $\frac{n^3 + 8}{n + 2}$

Solutions

1. a. $x^2 - x + 1$ b. $2 + 3x - 4y$ c. $x + 3 + \frac{1}{x}$

d. $\frac{a}{b} + 1$ e. $\frac{x}{y} - 1$ f. $a + b$

2. a. $x + 2$ b. $x + 3$ c. $x + 1$

d. $n - 4 + \frac{16}{n + 5}$ e. $2a - 11 + \frac{35}{a + 3}$ f. $3w - 15 + \frac{85}{w + 5}$

g. $3b - 4 + \frac{-3}{2b + 3}$ h. $3y - 15 + \frac{66}{y + 5}$ i. $5x + 4 + \frac{-3}{2x - 1}$

j. $x^2 - x + 1$ k. $n^2 + 2n + 4$ l. $a + 3$

3. a. $8x^2 + 5x + 3$ b. $4w + 1$ c. $5r - 3$

d. $9m + 1$

4. $x - 2 + \frac{3}{2x} - \frac{5}{2x^2}$

5. $x + 5 + \frac{34}{x - 5}$

6. $x^2 - 3x + 6 + \frac{-10}{x + 3}$

7. $x^3 - x^2 + x - 1$

8. $n^2 - 2n + 4$

"When one door closes, another opens; but we often look so long and so regretfully upon the closed door that we do not see the one which has opened for us."

- *Alexander Graham Bell*