REAL NUMBERS AND THEIR PROPERTIES

❑ *THE REAL NUMBERS*

You have seen many different kinds of numbers in this algebra course:

$7 \qquad 0 \qquad -9 \qquad 2.835 \qquad -\frac{15}{4} \qquad \frac{2}{3} \qquad \pi \qquad -\sqrt{2}$

As different as all these numbers may seem, they actually have one critical common characteristic: They can all be written as **decimal numbers**:

> Actually, all the finite decimals listed here can be viewed as infinite, repeating decimals. How? By realizing that each of them could have an unending supply of zeros attached to their back end. Thus, any finite decimal is really a repeating decimal.

$7 = 7.0$	A finite decimal
$0 = 0.0$	A finite decimal
$-9 = -9.0$	A finite decimal
2.835	A finite decimal
$-\frac{15}{4} = -3.75$	A finite decimal
$\frac{2}{3} = 0.66666\ldots$	An infinite, repeating decimal
$\pi = 3.14159265\ldots$	An infinite, non-repeating decimal
$-\sqrt{2} = -1.41421356\ldots$	An infinite, non-repeating decimal

To distinguish between the numbers that are decimals and numbers like $\sqrt{-9}$, which can never be written as a decimal, the term **real number** was given to the decimals, and the term **imaginary number** was given to numbers like $\sqrt{-9}$.

Hundreds of years ago, mathematicians thought it was obvious which numbers were real and which were imaginary. But this was a rather arrogant attitude. After all, to an Elementary Algebra student, a real number like $\sqrt{2}$ (which is an infinite, non-repeating decimal) may not seem "real" at all. On the other hand, imaginary numbers, for example $\sqrt{-1}$, seem very real to people (like electronics engineers) who use them every day. The bottom line is this: The terms *real* and *imaginary* are completely arbitrary -- one person's reality is another's imagination. But we're stuck with the terms, so we might as well learn them.

In summary, we call any number that can be written as a decimal a ***real number***. The set of real numbers is often denoted by writing $\mathbb{R}$.

The Real Numbers, $\mathbb{R}$, can be divided into two separate groups:

The Rationals – these are the repeating decimals (which includes the finite decimals).

The Irrationals – the decimals that are non-repeating.

Homework

1. Classify each number as **real** or **imaginary**:

a. 123	b. -42	c. 0
d. 2.3	e. $\sqrt{-8}$	f. $\sqrt{144}$
g. $-\sqrt{81}$	h. $\sqrt{10}$	i. -23.78
j. $-\pi$	k. $\sqrt{3}$	l. $\sqrt{-121}$
m. 0.239057	n. $2.787878...$	o. 3.092748526
p. $3.1428669...$	q. $\sqrt{8765}$	r. $\sqrt{-0.25}$
s. $-\sqrt{-25}$	t. $\sqrt{-(-71)}$	u. 10^6

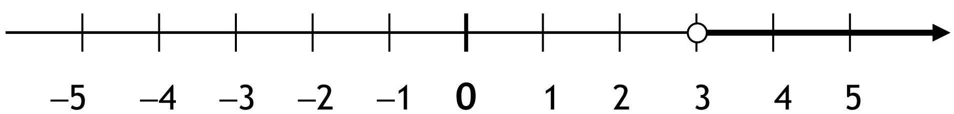
Now that we know what a real number is, we can analyze some of the properties of these real numbers. We are quite familiar with all of these properties; it's the official names of these properties that might cause us some grief.

❑ *DENOTING SUBSETS OF REAL NUMBERS*

First Example: Consider all the real numbers greater than 3. One simple way to express this set of numbers is the following:

$$x > 3$$

We can also graph this set of numbers on a number line:



Notice that we put an "open dot" at $x = 3$ to indicate the 3 is not part of the set of numbers. But the arrow goes infinitely to the right because $x > 3$ is the set of numbers greater than 3. Whether written as an inequality or a graph on a number line, note that the numbers 3.01, π, 17, and 200 are part of the set; but –5, 0, 2.5 and 3 are not part of the set.

And we have a third way, *interval notation*, to denote all the real numbers greater than 3. We write

$$(3, \infty)$$

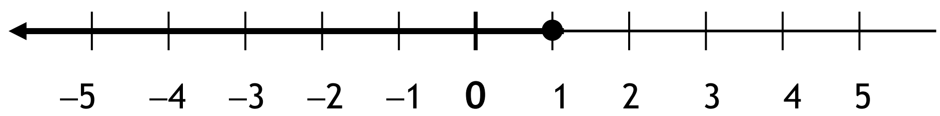
The two parentheses mean "exclude" when writing intervals.

You can read this as "all the real numbers between 3 and ∞, excluding the 3 (because 3 is not allowed), and excluding ∞, since ∞ isn't really a number you can get to; it's more of a concept than a number.

Second Example: Now we consider all the real numbers less than or equal to 1. This set of numbers can be written as an inequality like this:

$$x \leq 1$$

As a graph on a number line, we write:



The "solid dot" is used to indicate that $x = 1$ is part of the set of the numbers. And since x must be less than or equal to 1, the arrow goes infinitely to the left. Either as an inequality or a graph, you should see that the numbers -3, -1.1, $\frac{7}{8}$, and 1 are part of the set, while 1.001 and π and $\sqrt{2}$ are not part of the set.

In interval notation, this set is written

$(-\infty, 1]$ The square bracket means "include."

In this case, the square bracket on the 1 denotes the fact that 1 is in this set. And we always use a parenthesis on the $-\infty$, because it's not really a number.

Third Example: Sometimes a set of numbers is described in words like this: All real numbers except 7 and 11. A cool way to write this set is

$$\mathbb{R} - \{7, 11\}$$

This is read: The set of all real numbers, but with 7 and 11 removed.

❑ *THE CLOSURE PROPERTIES*

When a collie mates with a collie, it's quite certain that the offspring will be a collie. The same idea holds in the real numbers. For example, when two real numbers are added, the sum is a real number. Some examples:

$17 + 14.56 = 31.56$, which is a real number.

Let's add the real numbers π and $\sqrt{2}$. The sum is $\pi+\sqrt{2}$, which cannot be simplified, but I guarantee that the sum is a real number. Here's a little proof: Recall that a real number is any number which can be written as a decimal, so

$$\pi+\sqrt{2} = 3.1415926\ldots \qquad \ldots \qquad \ldots$$

which is certainly a decimal, and therefore a real number. Hence $\pi+\sqrt{2}$ is a real number.

We say that the set of real numbers is ***closed*** under addition. The set of real numbers is also closed under subtraction and multiplication. And with one critical exception, it's also closed under division. Do you know what that exception is?

❑ *THE COMMUTATIVE PROPERTIES*

Just as a governor can commute your death sentence to a life sentence by "switching" things around, we can switch, or reverse, the factors of 7×3 and write it as 3×7, and we've known since we were little tykes that they're the same product, 21.

More generally, the factors of the product xy can be commuted, or switched, and be written yx, without changing the answer. We say that multiplication is a ***commutative*** operation. Anytime we want to switch the order of two factors, we can simply do it.

Addition is also a commutative operation; after all, it's pretty clear that $x + y = y + x$ for any real numbers x and y.

Now it gets interesting: Is subtraction a commutative operation? That is, does $x - y = y - x$ for all value of x and y? Of course not; does $10 - 2 = 2 - 10$?

You decide whether division is a commutative operation.

❑ THE ASSOCIATIVE PROPERTIES

Consider the sum 5 + 3 + 2. We've learned, via the Order of Operations, that we can certainly work the problem left to right, which means we're going to add the 5 and 3 first; let's use parentheses around the 5 + 3:

$$(5 + 3) + 2 = 8 + 2 = 10$$

Now let's see if we get the same answer if we add the 3 and 2 first:

$$5 + (3 + 2) = 5 + 5 = 10$$

We get the same answer. It appears that $(5 + 3) + 2 = 5 + (3 + 2)$. That is, we can "shift" the parentheses left-to-right or right-to-left.

The same property holds for multiplication, too. For example,

$$(7 \times 5) \times 2 = 35 \times 2 = 70$$

and, $7 \times (5 \times 2) = 7 \times 10 = 70$

In summary, the ***associative properties*** state the following: For any real numbers x, y, and z:

$$\boldsymbol{(x + y) + z = x + (y + z)}$$

$$\boldsymbol{(xy)z = x(yz)}$$

❑ THE DISTRIBUTIVE PROPERTY

This property is the cornerstone of algebra, and we've used it constantly throughout the course. It is the only property in this chapter which involves two different operations, multiplication and addition.

For any real numbers a, b, and c:

$$\boldsymbol{a(b + c) = ab + ac}$$

❑ THE ADDITIVE IDENTITY

Would you agree that adding 0 to a real number results in that same number? In other words, that $\boldsymbol{x + 0 = x}$ for all real numbers x. Of course you would agree; you knew that when you were a little kid. Leave it to the math geeks to come up with a fancy term for something so simple. We call the number zero the ***additive identity*** because when zero is **added** to a number, the sum is **identical** to that number. There is only one additive identity in the set of real numbers. In other words, zero is the only real number that acts like zero.

❑ THE MULTIPLICATIVE IDENTITY

More silliness -- we know that any number times 1 is itself: $\boldsymbol{x \cdot 1 = x}$ for any real number x. This special number, 1, is given the name ***multiplicative identity*** (accent on the "plic"), because when a number is **multiplied** by 1, the product is **identical** to that number. And it should be clear that 1 is the only number that possesses this property.

❑ THE ADDITIVE INVERSE

Remember the term "opposite"? The opposite of 7 is –7, the opposite of –99 is 99, and the opposite of 0 is 0. The official term for "opposite" is ***additive inverse***, so now we can talk fancy and say things like, "The additive inverse of –23 is 23." Every real number has an additive inverse; in general, the additive inverse of n is $-n$. Also notice that when a number and its additive inverse are <u>added</u>, the <u>sum</u> is always 0, the additive identity. That is, for every real number n,

$$\boldsymbol{n + (-n) = 0}$$

❑ THE MULTIPLICATIVE INVERSE

Here's some more jargon, and it simply means "reciprocal." Thus, the multiplicative inverse of $\frac{3}{5}$ is $\frac{5}{3}$. In general, the multiplicative inverse of x is $\frac{1}{x}$, and the multiplicative inverse of $\frac{a}{b}$ is $\frac{b}{a}$. Does every real number have a multiplicative inverse? Absolutely NOT. Consider zero; if zero had a multiplicative inverse, it would be $\frac{1}{0}$, which is undefined. Therefore, every real number except zero has a multiplicative inverse. Notice that when a number and its multiplicative inverse are multiplied, the product is always 1, the multiplicative identity. In other words, for any real number x (other than 0),

$$\boldsymbol{x\left(\frac{1}{x}\right) = 1}$$

❑ SUMMARY OF THE PROPERTIES OF THE REAL NUMBERS $\mathbb{R}$

Closure Properties

If x and y are real numbers, then

$x + y$ is a real number

$x - y$ is a real number

xy is a real number

$\frac{x}{y}$ is a real number, provided $y \neq 0$

Commutative Properties

If x and y are real numbers, then

$$x + y = y + x$$

and $$xy = yx$$

Associative Properties

If x and y are real numbers, then

$$(x + y) + z = x + (y + z)$$

and

$$(xy)z = x(yz)$$

Distributive Property

If x, y, and z are real numbers, then

$$x(y + z) = xy + xz$$

Additive Identity

If x is a real number, then

$$x + 0 = x$$

Multiplicative Identity

If x is a real number, then

$$x \cdot 1 = x$$

Additive Inverse

If x is a real number, then there is a real number, denoted $-x$, such that

$$x + (-x) = 0$$

Multiplicative Inverse

If x is a nonzero real number, then there is a real number, denoted $\frac{1}{x}$, such that

$$x \cdot \frac{1}{x} = 1$$

Homework

2. a. The set of real numbers, $\mathbb{R}$, is **closed** under division, with one exception. What is that exception?

   b. Explain what it means to say that the set of real numbers is closed under multiplication.

3. Which operations (addition, subtraction, multiplication, division) are **commutative** operations in $\mathbb{R}$? Give an example which demonstrates which are commutative and which are not.

4. Prove that subtraction and division are NOT **associative** operations in $\mathbb{R}$.

5. What's the official term for the number 0 in $\mathbb{R}$?

6. What's the official term for the number 1 in $\mathbb{R}$?

7. Find the **additive inverse** of each real number:

   a. 17   b. -3 c. 0   d. $-\pi$

   e. $-\frac{7}{4}$ f. x g. $-n$

8. The <u>sum</u> of any real number and its additive inverse is ___.

9. Find the **multiplicative inverse** of each real number:

   a. 22   b. -9 c. 0

   d. $\frac{1}{\pi}$ e. $-\frac{\sqrt{3}}{\sqrt{2}}$ f. N

10. The <u>product</u> of any real number and its multiplicative inverse is ___.

11. a. T/F: Every real number has an opposite.

    b. T/F: Every real number has a reciprocal.

MATCHING:

12. __	$a(b + c) = ab + ac$	a. commutative
13. __	$a + b = b + a$	b. associative
14. __	xy is a real number if x and y are.	c. distributive
15. __	$-a$	d. closure
16. __	1	e. additive identity
17. __	$\frac{1}{a}$	f. additive inverse of a
18. __	$x(yz) = (xy)z$	g. multiplicative identity
19. __	0	h. multiplicative inverse of a
20. __	$uw = wu$	
21. __	$(a + b) + c = a + (b + c)$	
22. __	$a - b$ is a real number if a and b are real numbers	
23. __	$x + (y + z) = x + (z + y)$	
24. __	$a^2b^3 = b^3a^2$	
25. __	If w is a real number, so is w^6.	
26. __	$\sqrt{2} - \sqrt{3} + 10\pi$ is a real number.	
27. __	$(ab)(cd) = a(bc)d$	

Solutions

1.

a.	Real	b.	Real	c.	Real	d.	Real
e.	Imaginary	f.	Real	g.	Real	h.	Real
i.	Real	j.	Real	k.	Real	l.	Imaginary
m.	Real	n.	Real	o.	Real	p.	Real
q.	Real	r.	Imaginary	s.	Imaginary	t.	Real
u.	Real						

2. a. Dividing any number by 0 does <u>not</u> produce a real number.

b. It means that if x and y are any two real numbers, their product xy is also a real number. For example, since π and $\sqrt{7}$ are real numbers, so is their product $\pi\sqrt{7}$.

3. Addition and multiplication are commutative operations. For example,

$$1+\sqrt{2} = \sqrt{2}+1 \quad \text{and} \quad 100 \times \pi = \pi \times 100$$

Subtraction and division are NOT commutative operations. For instance,

$$34 - 10 \neq 10 - 34 \quad \text{and} \quad \frac{10}{2} \neq \frac{2}{10}$$ (Confirm these statements!)

4. If subtraction were associative, the following would have to be true:

$$10 - (7 - 4) = (10 - 7) - 4$$

$$\Rightarrow \quad 10 - 3 = 3 - 4$$

$$\Rightarrow \quad 7 = -1$$, which of course it isn't.

If division were associative, the following would hold:

$$20 \div (10 \div 2) = (20 \div 10) \div 2$$

The problem is, the left side equals 4, while the right side equals 1.

5. Additive identity

6. Multiplicative identity

7. a. -17 b. 3 c. 0 d. π e. $\frac{7}{4}$ f. $-x$ g. n

8. 0

9. a. $\frac{1}{22}$ b. $-\frac{1}{9}$ c. Does not exist d. π e. $-\frac{\sqrt{2}}{\sqrt{3}}$ f. $\frac{1}{N}$

10. 1

11. a. True b. False

12. c. **13**. a. **14**. d. **15**. f. **16**. g. **17**. h.

18. b. **19**. e. **20**. a. **21**. b. **22**. d. **23**. a.

24. a. **25**. d. **26**. d. **27**. b.

"It is harder to crack a prejudice than an atom."

– A. Einstein