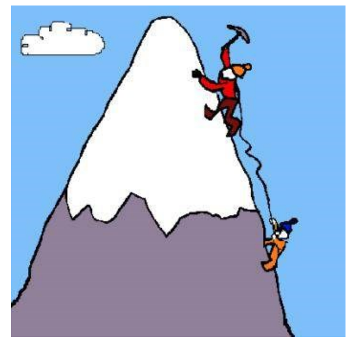
CH 38 – SLOPE

❑ INTRODUCTION



A line has many attributes, or characteristics. Two of the most important are its *intercepts* and its *slope*. The intercepts tell us where the line crosses the x-axis and the y-axis; they are very good reference points. The ***slope*** of a line tells us how *steep* the line is -- it's kind of like the angle that a line makes, and is a concept used in economics, chemistry, statistics, construction, and mountain climbing.

❑ SLOPE

A trucker is keenly aware of the *grade*, or angle, of the road on which a truck travels -- it determines the speed limit and the proper gear that the truck needs to be in. A roofer is concerned with the *pitch*, or steepness, of a roof. A construction worker needs to make sure that a wheelchair ramp has the correct *angle* with the street or sidewalk. All of these ideas are examples of the concept "*steepness*."

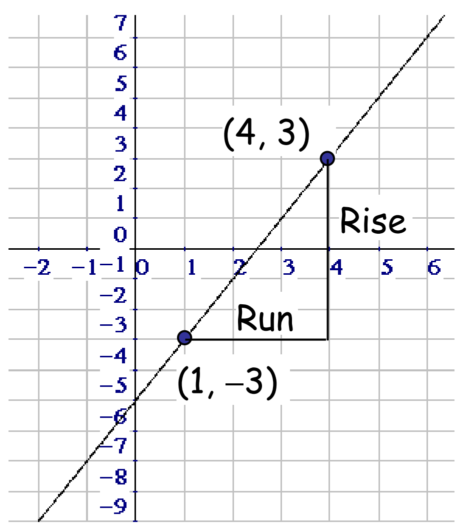
We'll use the term ***slope*** to represent steepness, and give it the letter m (I don't know why -- maybe m for mountain?). Our definition of slope in this course and all future math courses (and chemistry, economics, and nursing courses) is as follows:

$$m = \frac{\text{rise}}{\text{run}}$$

As we'll see shortly, a **rise** is a vertical (up/down) change, while a **run** is a horizontal (left/right) change. Slope is defined as the *ratio* of the rise to the run; we can also say that slope is the *quotient* of the rise and the run.

EXAMPLE 1: **Graph the line $y = 2x - 5$ and determine its slope.**

Solution: Let's calculate a couple of points by choosing some random x-values. If we let $x = 1$, then $y = -3$, so the point $(1, -3)$ is on the line. And if we let $x = 4$, then $y = 3$, giving us the point $(4, 3)$. We could calculate more points for our line, but let's cut to the chase and graph the line given the two points just computed.



Notice that we've constructed a right triangle using the line segment between the two given points as the hypotenuse. The rise and run are then just the lengths of the legs of the triangle. Counting squares from left to right along the bottom of the triangle, we see that the run is 3. Counting squares up the side of the triangle yields a rise of 6. Using the slope formula, we can calculate the slope of the line:

$$m = \frac{\text{rise}}{\text{run}} = \frac{6}{3} = \boxed{2}$$

Note: The concept of slope is dimensionless; that is, slope has no units. Here's why: Suppose that the units in the triangle are in feet. Then the slope is

$$m = \frac{\text{rise}}{\text{run}} = \frac{6\,\text{ft}}{3\,\text{ft}} = \frac{6\,\cancel{\text{ft}}}{3\,\cancel{\text{ft}}} = \mathbf{2} \text{ (since the feet cancel out)}$$

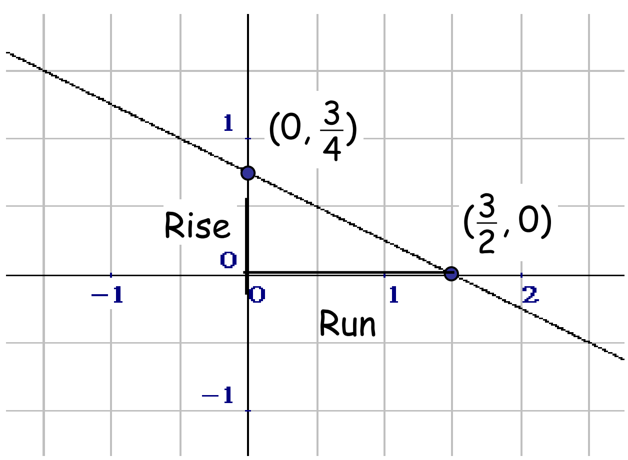
<u>EXAMPLE 2:</u> **Find the slope of the line $2x + 4y = 3$.**

<u>Solution:</u> To graph this line, let's calculate the two intercepts (since they're generally the easiest points to calculate). Set $x = 0$ to get

$$2(\mathbf{0}) + 4y = 3 \Rightarrow 4y = 3 \Rightarrow y = \frac{3}{4}$$

Thus, the y-intercept is $(0, \frac{3}{4})$. If we set $y = 0$, we can solve for x:

$2x + 4(\mathbf{0}) = 3 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$, which implies that the x-intercept is $(\frac{3}{2}, 0)$. Plotting these two intercepts gives us our line:



As we move from left to right, from the y-intercept to the x-intercept, we notice that the rise is actually a drop -- this means that the rise is negative. Since the height of the triangle is $\frac{3}{4}$, we conclude that the "rise" is $-\frac{3}{4}$. Since the run is from left to right, the run is positive $\frac{3}{2}$. Now we're ready for the calculation:

$$m = \frac{\text{rise}}{\text{run}} = \frac{-\frac{3}{4}}{\frac{3}{2}} = -\frac{3}{4} \div \frac{3}{2} = -\frac{3}{4} \cdot \frac{2}{3} = \boxed{-\frac{1}{2}}$$

Note: Instead of moving from left to right, from the y-intercept to the x-intercept, we could also have moved from right to left, from the x-intercept to the y-intercept. In this case, the rise is positive because we're moving up, but the run is negative because we're moving to the left. This will still give us the same answer, since now the calculation looks like:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\frac{3}{4}}{-\frac{3}{2}} = \frac{3}{4} \div -\frac{3}{2} = \frac{3}{4} \cdot -\frac{2}{3} = -\mathbf{\frac{1}{2}}$$

Homework

1. For each pair of points, plot them on a grid, find the rise and the run, and then use the formula for slope to calculate the *slope* of the line connecting the two points:

   a. (2, 3), (4, 7) b. (–3, 0), (0, 6) c. (1, –3), (–2, 5)

   d. (2, 2), (7, 7) e. (–3, –3), (0, 0) f. (–1, –2), (3, –5)

2. Find the *slope* of the given line by graphing the line and using the rise and run. You may, of course, use any two points on the line to calculate the rise and the run:

   a. $y = x + 3$ b. $y = 2x - 1$ c. $y = -2x + 3$

d. $y = 3x + 1$ e. $y = -3x - 2$ f. $y = -x + 2$

g. $x + 2y = 4$ h. $2x - 3y = 1$ i. $3x - y = 3$

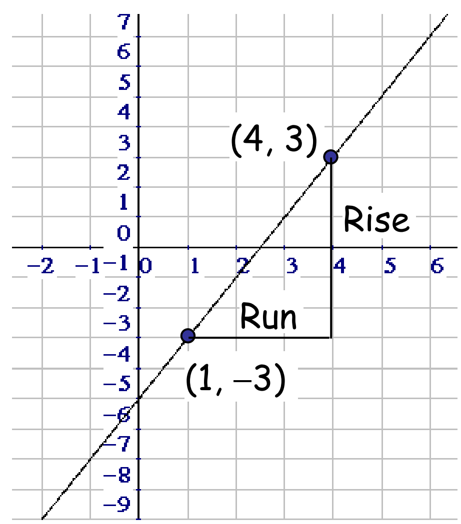
j. $-3x + 2y = 6$ k. $2x + 5y = 10$ l. $3x - 4y = -8$

❑ A NEW VIEW OF SLOPE

Finding the slope, $m = \frac{\text{rise}}{\text{run}}$, of a line by plotting two points and counting the squares to determine the rise and the run works fine only when it's convenient to plot the points and you're in the mood to count squares. Indeed, consider the line connecting the points (π, 2000) and (3π, –5000). Certainly these points determine a line, and that line has some sort of slope, but plotting these points is not really feasible -- we need a simpler way to calculate slope.

Recall Example 1 from this chapter, $y = 2x - 5$. We plotted the points (4, 3) and (1, –3) and then counted squares (as we moved from left to right) to get a rise of 6 and a run of 3, giving us a slope of

$$m = \frac{\text{rise}}{\text{run}} = \frac{6}{3} = \mathbf{2}$$



How can we get the numbers 6 and 3 <u>without</u> referring to the points on the graph? Notice that if we subtract the y-coordinate of one point from the y-coordinate of the other point, we get

$$\text{rise} = 3 - (-3) = 3 + 3 = 6$$

Similarly, if we subtract one x-coordinate from the other, we get

$$\text{run} = 4 - 1 = 3$$

Now dividing the rise by the run gets us our slope of 2. We can now think of our $m = \frac{\text{rise}}{\text{run}}$ formula as

$$\boxed{m = \frac{\text{change in } y}{\text{change in } x}}$$

The only issue we need to worry about is that we are consistent in the direction in which we do our subtractions. For instance, using the same two points, (1, –3) and (4, 3), we can subtract in the reverse order from above, as long as both subtractions are reversed.

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-3-3}{1-4} = \frac{-6}{-3} = \mathbf{2}$$

the same value of slope calculated before.

EXAMPLE 3: **Find the slope of the line connecting the points (–7, –13) and (12, –10). Calculate the slope again by subtracting in the reverse direction.**

Solution: Subtracting in one direction computes the slope as:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-13-(-10)}{-7-12} = \frac{-13+10}{-7-12} = \frac{-3}{-19} = \mathbf{\frac{3}{19}}$$

Reversing the direction in which we subtract the coordinates:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-10-(-13)}{12-(-7)} = \frac{-10+13}{12+7} = \mathbf{\frac{3}{19}}$$

Either way, we get the same slope; thus, the order in which you subtract is entirely up to you, as long as each subtraction (top and bottom) is done in the same direction.

New Notation

We're just about ready to find the slope of a line using the points mentioned at the beginning of this section: (π, 2000) and (3π, –5000). But first we introduce some new notation.

The natural world is filled with changes. In slope, we've seen changes in x and y in the notions of rise and run. In chemistry, there are changes in the volume and pressure of a gas. In nursing, there are changes in temperature and blood pressure, and in economics there are changes in supply and demand. This concept occurs so often that there's a special notation for a "change" in something. We use the Greek capital letter delta, Δ, to represent a change in something. A change in volume might be denoted by ΔV and a change in time by Δt. And so now we can redefine ***slope*** as

$$\boxed{m = \frac{\Delta y}{\Delta x}}$$

Slope is the *ratio* of the change in *y* to the change in *x*.

which is, of course, just fancy notation for what we already know.

<u>EXAMPLE 4:</u> **Find the slope of the line connecting the points (π, 2000) and (3π, –5000).**

<u>Solution:</u> A simple ratio calculation will give us the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{2000-(-5000)}{\pi - 3\pi} = \frac{7000}{-2\pi} = \frac{\cancel{2}\cdot 3500}{-\cancel{2}\pi} = -\frac{\mathbf{3500}}{\pi}$$

In the last step of this calculation we used the fact that a positive number divided by a negative number is negative. Also, we could obtain an approximate answer by dividing 3500 by 3.14 -- then attaching the negative sign -- to get about –1,114.65.

Notice that there's no need to plot points and count squares on a grid. We've turned the geometric concept of slope into an arithmetic problem. Try reversing the order of the subtractions above to make sure you get the same slope.

Δ

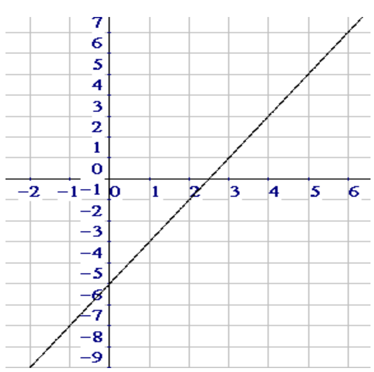
Why use delta, Δ, to represent a change in something? Because "delta" begins with a *d*, and *d* is the first letter of the word "difference," and difference means "subtract," and subtract is what you do when you want to calculate the change in something.

Homework

3. Use the formula $m = \dfrac{\Delta y}{\Delta x}$ to find the slope of the line connecting the given pair of points:

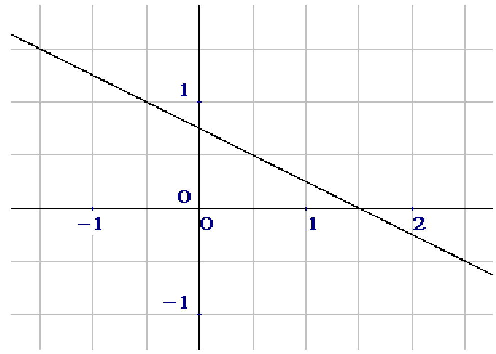
a. (2, 3) and (4, 7)
b. (–3, 0) and (0, 6)
c. (1, –3) and (–2, 5)
d. (2, 2) and (7, 7)
e. (–3, –3) and (0, 0)
f. (–1, –2) and (3, –5)
g. (1, 1) and (–2, 3)
h. (1, 4) and (0, 0)
i. (–3, –2) and (1, –3)
j. (–1, 3) and (1, –3)
k. (–4, 5) and (0, 0)
l. (–1, –1) and (4, –2)

❑ *THE SLOPES OF INCREASING AND DECREASING LINES*



Looking back at Example 1 of this chapter, let's make a quick sketch of the line. We can call this an "increasing" line, because as we move from left to right, the line is rising, or increasing, since the y-values are getting bigger. Now notice that the slope of this line, as calculated before, was 2, a <u>positive</u> number.

Referring now to Example 2, we find that its graph, unlike the previous one, is falling as we move from left to right -- that is, we have a "decreasing" line. And this is due to the fact that the y-values are getting smaller. Next we note that the slope was calculated to be the negative number $-\frac{1}{2}$.



This connection between the "increasing/decreasing" of a line and the sign of its slope is always true. Our conclusion is the following:

> An increasing line has a positive slope, while a decreasing line has a negative slope.

Review Problems

4. Find the slope of the line connecting the given pair of points. Use the slope to determine whether the graph of the line is increasing or decreasing.

   a. (–10, 7) and (–12, –8)   b. (12, –10) and (8, –5)

   c. (12, 3) and (–3, 10)   d. (1, 3) and (10, 5)

   e. (–8, 10) and (12, 8)   f. (–9, 1) and (–10, 11)

g. (–2, –1) and (1, 5)   h. (6, –1) and (–12, –1)

i. (4, 6) and (9, –5)   j. (3, –3) and (12, 6)

Solutions

1. a. 2 b. 2 c. $-\frac{8}{3}$ d. 1 e. 1 f. $-\frac{3}{4}$

2. a. 1 b. 2 c. –2 d. 3 e. –3 f. –1
g. $-\frac{1}{2}$ h. $\frac{2}{3}$ i. 3 j. $\frac{3}{2}$ k. $-\frac{2}{5}$ l. $\frac{3}{4}$

3. a. 2 b. 2 c. $-\frac{8}{3}$ d. 1 e. 1 f. $-\frac{3}{4}$
g. $-\frac{2}{3}$ h. 4 i. $-\frac{1}{4}$ j. –3 k. $-\frac{5}{4}$ l. $-\frac{1}{5}$

4. If the slope is positive, the line is increasing; if the slope is negative, the line is decreasing. But what about part h. of this problem?

a. $\frac{15}{2}$ b. $-\frac{5}{4}$ c. $-\frac{7}{15}$ d. $\frac{2}{9}$ e. $-\frac{1}{10}$
f. –10 g. 2 h. 0 i. $-\frac{11}{5}$ j. 1

“Human history becomes more and more a race between education and catastrophe.”

– H.G. Wells (1866-1946)