Ch 44 – Inequalities

❑ Introduction

You must score between 80% and 89% to get a B in your math class.

You must be at least 18 years of age to vote.

You can be no taller than 48 inches to play in the park.

These are all examples of quantities being greater than something or less than something (or between two somethings). Since they are not equalities, they are called <u>in</u>equalities.

❑ Notation

We know that 5 is bigger than 3, which we can write as "**5 > 3**." The symbol ">" can also be read as "is larger than" or "is greater than."

> \> means "is greater than"
>
> < means "is less than"
>
> ≥ means "is greater than or equal to"
>
> ≤ means "is less than or equal to"

But, of course, the fact that 5 is larger than 3 is the same as the fact that 3 is less than 5. This is written "**3 < 5**."

The symbol "≥" can read "is greater than or equal to." For example, **9 ≥ 7** because 9 is indeed greater than or equal to 7. (Actually, it's greater than 7, but that doesn't change the fact that it's greater than <u>or</u> equal to 7.)

Believe it or not, **$-12 \geq -12$** is also a true statement -- after all, since -12 equals -12, it's certainly the case that -12 is greater than or equal to -12.

The symbol "$\leq$" is read "is less than or equal to." A couple of examples are $6 \leq 10$ and $8 \leq 8$.

Homework

1. True/False:

   a. $7 > 3$ b. $-2 < 1$ c. $13 \geq 13$ d. $-9 \leq -9$

   e. $12 \geq 9$ f. $-18 \leq -20$ g. $\pi > 0$ h. $-\sqrt{2} \leq 0$

2. Express each statement as an inequality:

   a. Your age, a, must be at least 18 years.

   b. Your height, h, can be no taller than 48 inches.

   c. Your years of experience, y, must exceed 10 years.

   d. The number of driving tickets, t, must be fewer than 5.

   e. The mean, μ (Greek letter mu), must be at least 75.

   f. The standard deviation, σ (Greek letter sigma), must be no more than 10.

   g. The energy, E, must be more than 100.

   h. The mass, m, must be less than 3.7.

❑ The Basic Principle of Solving Inequalities

To solve an inequality such as $-2x + 7 \geq -4$, we have to perform the operations necessary to isolate the x. We will accomplish this goal by using the standard *do the same thing to each side* rule, but since we're talking about an inequality -- not an equation -- we have to be very careful. The following experiment should illustrate the potential problems, and give us the techniques to surmount these problems.

Let's perform six experiments on a given inequality. Consider the true statement

$$4 < 6$$

i. Add 10 to each side: $4 + \mathbf{10} < 6 + \mathbf{10}$

$\Rightarrow 14 < 16$ ✓

ii. Subtract 3 from each side: $4 - \mathbf{3} < 6 - \mathbf{3}$

$\Rightarrow 1 < 3$ ✓

iii. Multiply each side by 5: $4(\mathbf{5}) < 6(\mathbf{5})$

$\Rightarrow 20 < 30$ ✓

iv. Divide each side by 2: $\frac{4}{2} < \frac{6}{2}$

$\Rightarrow 2 < 3$ ✓

v. Multiply each side by –3: $4(-3) < 6(-3)$

$\Rightarrow -12 < -18$ ✗

vi. Divide each side by –2: $\frac{4}{-2} < \frac{6}{-2}$

$\Rightarrow -2 < -3$ ✗

What can we deduce (conclude) from these six calculations? The first two show that adding the same number to each side of an inequality, or subtracting the same number from each side of an inequality, are both things we can do without any issues; they lead to a true inequality.

The next two calculations indicate that multiplying or dividing each side of an inequality by a positive number always leads to an inequality which is just as valid as the original one.

Unfortunately, the last two cases -- multiplying or dividing by a negative number -- have led to false statements. So in these two scenarios, we must reverse the order of the inequality sign (flip it around) in order to maintain a true statement. Problem solved!

These numerical experiments are by no means a complete proof of the principle we are about to state, but they're convincing enough for me.

The Basic Principle of Inequalities

If you multiply or divide each side of an inequality by a *negative* number, you must reverse the inequality symbol.

EXAMPLE 1: **Solve each inequality:**

A. Solve the inequality: $\boldsymbol{x + 3 > 4}$

Subtract 3 from each side $\boldsymbol{x > 1}$

Note: The inequality symbol was not reversed.

B. Solve the inequality: $\boldsymbol{-2n - 9 \le 13}$

Add 9 to each side: $-2n \le 22$
(The inequality symbol was not reversed.)

Divide each side by -2: $\boldsymbol{n \ge -11}$

This time the inequality symbol was reversed, since we divided each side of the inequality by a negative number.

C. Solve the inequality: $\frac{u}{5}+3 < -4$

Subtract 3 from each side: $\frac{u}{5} < -7$

Multiply each side by 5: $u < -35$

Neither operation required reversing the inequality.

Homework

3. Solve each inequality:

a. $x + 7 > -10$ b. $x - 3 \le 5$ c. $2x \ge 14$

d. $-3x < -42$ e. $\frac{x}{8} > -3$ f. $\frac{x}{-5} \ge 10$

4. Solve for y: $6(y - 5) - (y + 1) \le 12y + 21$

5. Solve for u: $-2(7 + 3u) - (1 - u) > 2u - 10$

6. a. Solve for x: $\frac{-3x+5}{-2}+9 > 16$ Note: The process is more important than the answer. Do you know what this means?

b. Solve for a: $\frac{7a-5}{-3}-12 < 12$

c. Solve for y: $\frac{9-y}{-2} \ge -2$

d. Solve for n: $\frac{-14-2n}{12} \le -1$

❑ Double Inequalities

Suppose Edwin wants to take part in a political poll. The pollster only cares about voters whose age, a, is between 25 and 40 (including the 25 and the 40). Here's one way to write this requirement using algebra:

$a \geq 25$ **and** $a \leq 40$ (a double inequality using **and**)

By the way, the word **and** between the two inequalities is crucial; Edwin must be at least 25 **and** 40 or younger. Do you see that replacing the **and** with **or** would change the meaning entirely?

Here's the other way to write that double inequality -- we sandwich the variable a between the lower limit (25) and the upper limit (40):

$$\mathbf{25 \leq a \leq 40}$$

This can be read in two ways:

1. $a \geq 25$ **and** $a \leq 40$ [Note that $a \geq 25$ is the same as $25 \leq a$.]
2. a is *between* 25 and 40, including the 25 and including the 40.

For a second example, the phrase "*x must be between 0 and 10, but not equal to either 0 or 10*" is written

$$\mathbf{0 < x < 10}$$

Homework

7. True/False:
   a. 7 satisfies the double inequality: $0 < x < 9$.
   b. 0 satisfies the double inequality: $0 < x < 9$.
   c. 10 satisfies the double inequality: $-5 \leq x \leq 10$.
   d. 0 satisfies the double inequality: $0 \leq x < \pi$.
   e. 2π satisfies the double inequality: $0 < x < 7$.

f. 100 satisfies the double inequality: $100 \le x < 101$.

8. a. Name the only solution of the double inequality

$$23 \le x \le 23$$

b. Find all values of x which satisfy the double inequality

$$50 < x < 50$$

9. Express each situation as a *double inequality*:

a. The weight, w, must be between 20 lb and 50 lb, including the 20 and the 50.

b. The score, s, must be between 100 pts and 300 pts, excluding the 100 and the 300.

c. The commission, C, must be between \$2000 and \$5750, inclusive of the endpoints.

d. The distance, d, must be between 123 m and 250 m, exclusive of the endpoints.

Solutions

1. a. T b. T c. T d. T e. T f. F g. T h. T

2. a. $a \ge 18$ b. $h \le 48$ c. $y > 10$ d. $t < 5$ e. $\mu \ge 75$ f. $\sigma \le 10$ g. $E > 100$ h. $m < 3.7$

3. a. $x > -17$ b. $x \le 8$ c. $x \ge 7$ d. $x > 14$ e. $x > -24$ f. $x \le -50$

4. $y \ge -\frac{52}{7}$

5. $u < -\frac{5}{7}$

6. a. $x > \frac{19}{3}$ b. $a < -\frac{67}{7}$ c. $y \geq 5$ d. $n \geq -1$

7. a. True b. False c. True

d. True e. True f. True

8. I'd like to know what you think.

9. a. $20 \leq w \leq 50$ b. $100 < s < 300$

c. $2000 \leq C \leq 5750$ d. $123 < d < 250$

❑ To ∞ and Beyond

1. Determine the values of *a* which satisfy the compound inequality:

$$a \geq 25 \text{ **or** } a \leq 40$$

2. Marty was trying to solve the inequality

$$ax + b > c$$

for *x*, and wrote the following:

$$ax + b > c$$

$$\Rightarrow \quad ax > c - b$$

$$\Rightarrow \quad x > \frac{c-b}{a}$$

Explain the fallacy (faulty reasoning) in Marty's reasoning.