CH 50 – INTRO TO EXPONENTS

❑ Introduction

We've seen an exponent of 2, **squaring**, a couple of times in this class: The area of a circle is given by the formula $A = \pi r^2$, and the Pythagorean Theorem has squaring in three different places. Even my favorite formula in the whole world, Albert Einstein's $E = mc^2$, contains squaring.

Some formulas involve **cubing**. For example, the volume of a cube, where s is the length of a side, is given by $V = s^3$ (which is why raising to the third power is called *cubing*). Even the unit "cubic meter" (the volume of a box that's one meter on all sides) is often written m^3.

Larger exponents are possible; consider the following examples:

$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = \mathbf{81}$

[4 factors of 3]

$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = \mathbf{32}$

[5 factors of 2]

$1^{10} = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = \mathbf{1}$

[10 factors of 1]

$(-4)^6 = (-4)(-4)(-4)(-4)(-4)(-4) = \mathbf{4096}$

[6 factors of –4]

$(-2)^7 = (-2)(-2)(-2)(-2)(-2)(-2)(-2) = \mathbf{-128}$

[7 factors of –2]

Homework

1. Evaluate each expression:

a. the square of 7
b. the cube of 5
c. the cube of 3
d. the square of 20
e. the square of –3
f. the cube of –4
g. the square of 1
h. the cube of 0
i. the cube of 1
j. the square of 0
k. the cube of 10
l. the square of –10
m. 3^5
n. $(-7)^2$
o. 1000^2
p. $(-2)^4$
q. $(-1)^7$
r. $(-2)^5$
s. $(-1)^{20}$
t. 10^6
u. 5^4
v. 1^{12}
w. 2^8
x. 0^{34}
y. $(-3)^4$
z. $(-3)^5$

❑ Strange Exponents

Let's use the base of 2 to discuss some more exponents. We know what an exponent of 2 or higher means. These examples should be familiar:

$2^2 = 4$ $2^4 = 16$ $2^8 = 256$

But numbers like 2^1, 2^0, and 2^{-3} might seem a bit strange (or extremely strange!). Let's attack each kind of number one at a time.

An Exponent of One

What do we mean by 2^1? Since the exponent tells us how many times to use the base as a factor (for example, 2^{10} represents 10 factors of 2), it follows that $2^1 = 2$, and that's it.

An Exponent of Zero

Now for the interesting exponent of zero -- what in the world could 2^0 possibly mean? If your first instinct is 0, then I might agree with you -- but we'd both be wrong! Let's make a list of known ***powers of* 2**, look for a pattern, and then extend that pattern to figure out what 2^0 is.

$$2^4 = 16$$
$$2^3 = 8$$
$$2^2 = 4$$
$$2^1 = 2$$

The exponents in the first column are clearly decreasing by 1 at each step, and each number in the second column is one-half of the number above it.

Look at the pattern in the powers of 2 in the first column: The exponents are decreasing by 1 each time -- the next power of 2 in the sequence ought to be 2^0. Now look at the sequence of numbers 16, 8, 4, and 2 in the second column. Each number is one-half the preceding number; that is, we're dividing by 2 at each step. Therefore, the next number in the sequence should be 1 (which is half of 2). So, continuing the two patterns extends the list above to the following:

$$2^4 = 16$$
$$2^3 = 8$$
$$2^2 = 4$$
$$2^1 = 2$$
$$\mathbf{2^0 = 1}$$

This leads to the conclusion that $2^0 = 1$ -- a very strange result, indeed.

$$\boxed{2^0 = 1}$$

We can now evaluate expressions like the following:

$$2^0[2^0 + (-2)^1 + 2^2 + (-2)^3 - (-2)^4]$$

$$= \quad 1\,[1 + (-2) + 4 + (-8) - (+16)]$$

$= \quad 1[1 - 2 + 4 - 8 - 16]$

$= \quad 1[-21]$

$= \quad \mathbf{-21}$

Homework

2. Evaluate each expression:

a. $2^0 + 2^0$

b. $2^0 \cdot 2^0$

c. $2^0 + 2^1 + 2^2 + 2^3 + 2^4$

d. $(1 + 1)^0$

e. $2^5 - 2^3 + 2^1 - 2^0$

f. $(8 - 6)^0 + (10 - 8)^1$

g. $2^0 \times 2^1 \times 2^2 \times 2^3 \times 2^4$

h. $\left(\frac{12}{6}\right)^0 + \left(\frac{100}{50}\right)^0 - (20 - 15 - 3)^0 + \left(3^2 - 7\right)^1$

Negative Exponents

Recall the powers of 2 we wrote above:

$2^4 = 16$

$2^3 = 8$

$2^2 = 4$

$2^1 = 2$

$2^0 = 1$

Remember the patterns we observed in the above list? The exponents decrease by 1 each time, and the answers are divided by 2 each time. The next exponent in the sequence is –1, and the next answer must be 1 ÷ 2, which equals $\frac{1}{2}$. We've reached the conclusion that

$$\boxed{2^{-1} = \frac{1}{2}}$$

The next exponent would be –2, and the next answer would be $\frac{1}{2}$ divided by 2, which is $\frac{1}{2} \div 2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. So now we see that

$$2^{-2} = \frac{1}{4}$$

Similarly, $2^{-3} = \frac{1}{8}$, $2^{-4} = \frac{1}{16}$, and so on.

Do you see what's really going on here? Can you predict what 2^{-10} is? Let's look closely at the fact that $2^{-4} = \frac{1}{16}$. Note that 16 is 2^4, meaning that we can write

$$2^{-4} = \frac{1}{2^4}$$

From this observation, it now should be reasonable to say that

$$2^{-10} = \frac{1}{2^{10}} = \frac{1}{1024}$$

Let's put everything we've seen into one big list:

$$\begin{aligned}
2^4 &= 16 \\
2^3 &= 8 \\
2^2 &= 4 \\
2^1 &= 2 \\
2^0 &= 1 \\
2^{-1} &= \frac{1}{2^1} = \frac{1}{2} \\
2^{-2} &= \frac{1}{2^2} = \frac{1}{4} \\
2^{-3} &= \frac{1}{2^3} = \frac{1}{8} \\
2^{-4} &= \frac{1}{2^4} = \frac{1}{16} \\
&\vdots \\
2^{-10} &= \frac{1}{2^{10}} = \frac{1}{1024} \\
&\vdots
\end{aligned}$$

Powers of 2

We can try to describe the effect of a negative exponent this way:

2 *raised to a negative exponent is the fraction*:
1 *over* 2 *raised to the positive exponent.*

As an algebraic formula, we may write

$$\boxed{2^{-n} = \frac{1}{2^n}} \quad \text{where } n = 1, 2, 3, \ldots$$

Note: The base 2 raised to a negative exponent is a positive number -- it may be small, but it's positive. Therefore,

2 raised to <u>any</u> power is always positive.

Homework

3. Evaluate each power of 2:

a. 2^5	b. 2^1	c. 2^0	d. 2^7	e. 2^8
f. 2^{-5}	g. 2^{-8}	h. 2^{-3}	i. 2^{-7}	j. 2^{-9}
k. 2^6	l. 2^{-1}	m. 2^{-4}	n. $2^0 + 2^1 + 2^2 + 2^{-1} + 2^{-2}$	

4. a. Explain why 1 is actually a power of 2.
   b. T/F: Every power of 2 is even.
   c. T/F: Every even number is a power of 2.
   d. T/F: 64 is a power of 2.
   e. T/F: 50 is a power of 2.

❑ Lauren's Allowance

Lauren tells her father that she will give up her allowance for the rest of her life -- and will pay for all of her college and graduate school costs herself -- if he will agree to the following plan: He gives her 2¢ on the first day of the upcoming month, 4¢ on the second day of the month, 8¢ on the third day of the month, 16¢ on the fourth day of the month, and so on until the 30th day of the month. Dad (who was a history major) thinks this is a great money-saving idea (after just 30 days, he's off the hook for allowance and for college), and accepts Lauren's proposal.

Day	# of Pennies
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1,024
⋮	⋮
30	1,073,741,824 (almost $11 million)

In the chart we have calculated Lauren's earnings in pennies for each of the first 10 days of the month; then we cut to the chase and calculated the number of pennies for the 30th day. Take your calculator and extend the table for days 11 through 29 and prove that the amount of money for day 30 is correct.

Now let's come up with a direct formula that computes the money earned from knowing just the day of the month, <u>without having to know all the previous days' amounts</u>. Notice that each amount of money is simply 2 raised to the power of the day. For example, consider the 9th day. If we raise 2 to the 9th power (use the exponent button on your calculator), we get 512. That is, $2^9 = 512$. Now calculate 2^{30} and you should get 1,073,741,824.

Who's smarter, Lauren or her father?

Homework

5. a. How many pennies will Lauren earn on the 15th day of the month?

   b. How many pennies will Lauren earn the 31st of the month if Dad agreed to extend the plan that far?

   c. On which day of the month did Lauren earn half of what she earned on the 30th day of the month?

6. Now let's reverse the question. I'll give you a penny amount, and you tell me which day of the month Lauren earned that amount of money.

   a. 512¢ b. 4,096¢ c. 1,048,576¢ d. 33,554,432¢

Review Problems

7. Evaluate each expression:

   a. the square of 5 b. the cube of –5 c. the square of –6

   d. 4^5 e. $(-2)^6$ f. $(-3)^3$

   g. $2^1 - 2^0 + 2^3 - 2^2 - (10-8)^0$

   h. $2^1 \times 2^2 \times 2^0$

   i. $\left(\frac{18}{9}\right)^0 \times \left(\frac{-50}{-25}\right)^1 + (1024 - 1022)^0 - (10^2 - 98)^1$

   j. 2^{-6} k. 2^{-1} l. 2^{-9} m. $2^{-1} + 2^{-2} + 2^{-3}$

8. True/False:

   a. 1 is a power of 2.
   b. 128 is a power of 2.
   c. 24 is a power of 2.

Solutions

1. a. 49 b. 125 c. 27 d. 400
e. 9 f. –64 g. 1 h. 0
i. 1 j. 0 k. 1000 l. 100
m. 243 n. 49 o. 1,000,000 p. 16
q. –1 r. –32 s. 1 t. 1,000,000
u. 625 v. 1 w. 256 x. 0
y. 81 z. –243

2. a. 2 b. 1 c. 31 d. 1
e. 25 f. 3 g. 1024 h. 3

3. a. 32 b. 2 c. 1 d. 128 e. 256
f. $\frac{1}{32}$ g. $\frac{1}{256}$ h. $\frac{1}{8}$ i. $\frac{1}{128}$ j. $\frac{1}{512}$
k. 64 l. $\frac{1}{2}$ m. $\frac{1}{16}$ n. $7\frac{3}{4}$

4. a. Since $1 = 2^0$, 1 is a power of 2.
b. False, since 1 is a power of 2.
c. False, since 6 is even but it's not a power of 2.
d. True
e. False

5. a. 32,768 b. 2,147,483,648 c. 29th of the month

6. a. day 9 b. day 12 c. day 20 d. day 25

7. a. 25 b. -125 c. 36 d. 1024 e. 64
f. -27 g. 4 h. 8 i. 1 j. $\frac{1}{64}$
k. $\frac{1}{2}$ l. $\frac{1}{512}$ m. $\frac{7}{8}$

8. a. T b. T c. F

"Being defeated is often a temporary condition. Giving up is what makes it permanent."

– *Marilyn Vos Savant*