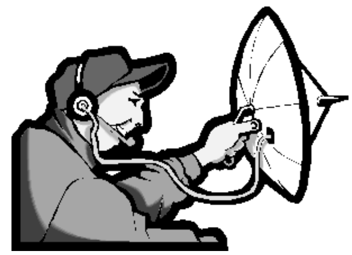
CH 80 – THE PARABOLA

❑ *INTRODUCTION*

The **parabola** (accent on the *rab*) is a very special shape used in searchlights and satellite dishes. Even football sports reporters use parabolic reflectors to listen in on comments made by coaches on the sidelines and players in the huddle. In fact, when a football is thrown or punted, its path is that of a parabola.



❑ *GRAPHING A PARABOLA*

EXAMPLE 1: **Graph:** $y = x^2 - 4x + 3$

Solution: Who's to say that the graph of this formula isn't a straight line? Let's work it out; we'll find some points on our graph by choosing some values of x, and then calculate the corresponding y-values -- and we'll see what points we get.

If $x = -1$, then $y = (\mathbf{-1})^2 - 4(\mathbf{-1}) + 3 = 1 + 4 + 3 = 8 \Rightarrow \mathbf{(-1, 8)}$

If $x = 0$, then $y = (\mathbf{0})^2 - 4(\mathbf{0}) + 3 = 0 - 0 + 3 = 3 \Rightarrow \mathbf{(0, 3)}$

If $x = 1$, then $y = (\mathbf{1})^2 - 4(\mathbf{1}) + 3 = 1 - 4 + 3 = 0 \Rightarrow \mathbf{(1, 0)}$

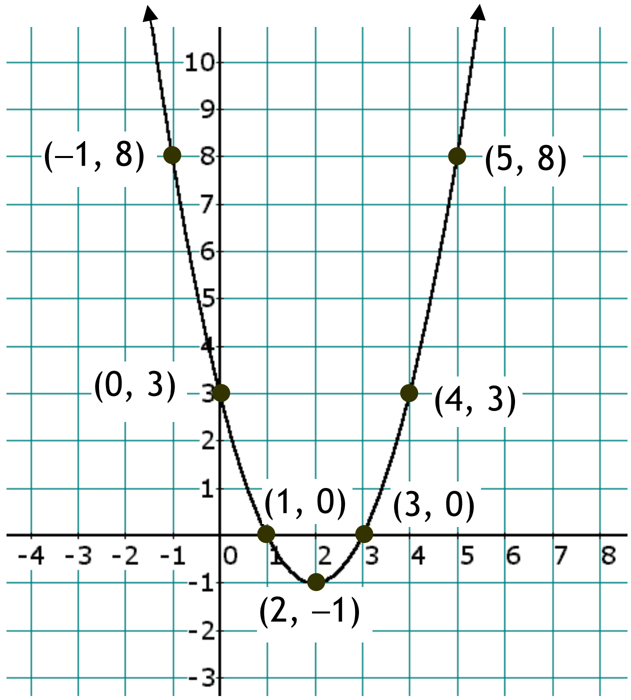
If $x = 2$, then $y = (\mathbf{2})^2 - 4(\mathbf{2}) + 3 = 4 - 8 + 3 = -1 \Rightarrow \mathbf{(2, -1)}$

If $x = 3$, then $y = (\mathbf{3})^2 - 4(\mathbf{3}) + 3 = 9 - 12 + 3 = 0 \Rightarrow \mathbf{(3, 0)}$

If $x = 4$, then $y = (\mathbf{4})^2 - 4(\mathbf{4}) + 3 = 16 - 16 + 3 = 3 \Rightarrow \mathbf{(4, 3)}$

If $x = 5$, then $y = (\mathbf{5})^2 - 4(\mathbf{5}) + 3 = 25 - 20 + 3 = 8 \Rightarrow \mathbf{(5, 8)}$

Plotting these seven points leads us to the following graph:



What do we notice about the graph? It's a curve, not a straight line. We notice that x can be any real number (but notice that y never goes below −1). Also note that the graph has one y-intercept but two x-intercepts. In addition, there is no highest point on the parabola, but the lowest point on the parabola is (2, −1), and we call this point the ***vertex*** of the parabola.

This is the shape called the ***parabola***. We say that the parabola just graphed "opens *up*." The equation of a parabola is characterized by the fact that one variable (the x) is *squared* while the other variable (the y) is raised to the first power.

EXAMPLE 2: **Graph:** $y = -x^2 - 2x - 1$

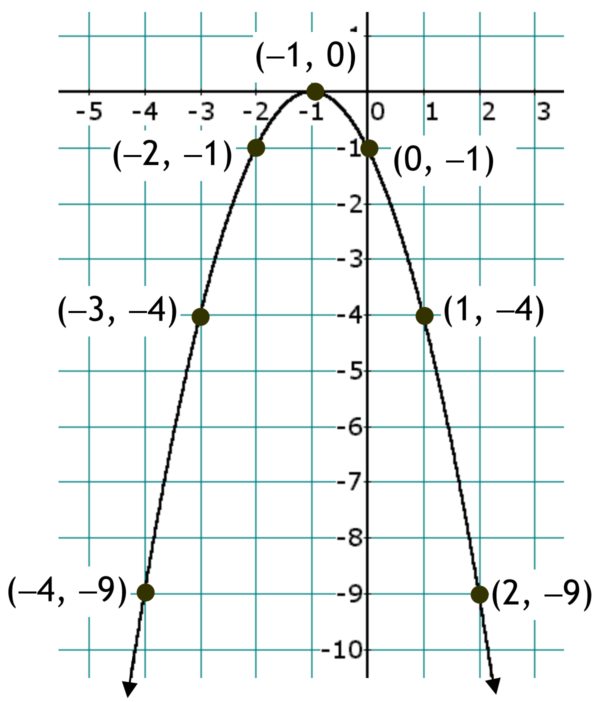
Solution: First we notice that the quadratic term, the $-x^2$, has a leading negative sign. And we remember, due to the Order of Operations, that exponents have a higher priority than minus signs, so we know that to evaluate $-x^2$, we square the x first, and apply the minus sign <u>second</u>. We'll calculate together two points on the parabola and leave the rest of the points for you to do.

If $x = 2$, then $y = -(\mathbf{2})^2 - 2(\mathbf{2}) - 1 = -4 - 4 - 1 = -9$. Thus, the point **(2, –9)** is on the graph of our parabola.

If $x = -3$, then $y = -(\mathbf{-3})^2 - 2(\mathbf{-3}) - 1 = -9 + 6 - 1 = -4$. Therefore, the point **(–3, –4)** is on the graph.

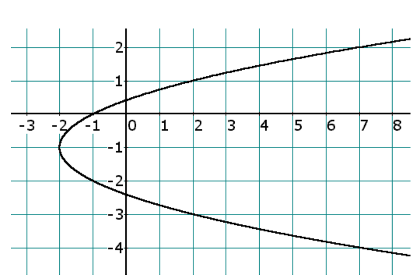
You should now do the calculations to show that each of the following points is also on the graph:

(–4, –9)   (–2, –1)   (–1, 0)   (0, –1)   (1, –4)



From the formula (and kind of from the graph) we note that *x* can be <u>any</u> real number, but that *y* never gets above 0. As for intercepts, there's one *x*-intercept and one *y*-intercept. There is no lowest point on the graph, but the highest point, the **vertex**, is the point (–1, 0). We say that this parabola opens *down*.

There are also "sideways" parabolas, but in this chapter only parabolas which open up or down will be discussed.



Homework

1. In Example 1 we saw that the graph of $y = x^2 - 4x + 3$ is a parabola opening <u>up</u>. Example 2 showed us that the graph of $y = -x^2 - 2x - 1$ is a parabola opening <u>down</u>. Take a guess what property of these equations determines whether the parabola opens up or down.

2. True/False: (Recall that all parabolas in this chapter open up or down.)

   a. Every parabola has a y-intercept.

   b. Every parabola has an x-intercept.

   c. Every parabola has a vertex.

   d. The vertex of a parabola is always the highest point of the parabola.

   e. The vertex of a parabola is always the lowest point of the parabola.

3. Graph each parabola by plotting points. Then use your graph to determine the intercepts and the vertex of your parabola:

   a. $y = 9 - x^2$

   b. $y = x^2 + 6x + 5$

   c. $y = x^2 + 2x - 8$

   d. $y = -x^2 + 4x - 4$

   e. $y = 0.5x^2$

   f. $y = -0.2x^2 + 5$

❑ *FINDING INTERCEPTS*

Remember the method for finding the intercepts of a line? Well, an intercept is an intercept, so the rules for finding the intercepts of a parabola (or any graph at all) are identical to the rules we learned before:

> x-intercepts are found by setting y to 0.
>
> y-intercepts are found by setting x to 0.

Also recall that an intercept is a point in the plane, and should be written as an ordered pair like (2, 0) or (0, –3). Ask your teacher if you need to write your intercepts this way.

EXAMPLE 3: **Find the intercepts of $y = x^2 - x - 6$.**

Solution:

x-intercepts: Setting $y = 0$

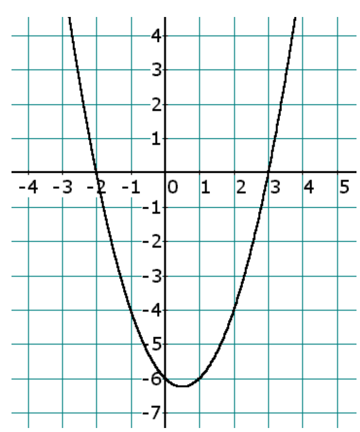
$\Rightarrow \mathbf{0} = x^2 - x - 6$

$\Rightarrow 0 = (x + 2)(x - 3)$

$\Rightarrow x + 2 = 0$ or $x - 3 = 0$

$\Rightarrow x = -2$ or $x = 3$ We conclude that

> the x-intercepts are (–2, 0) and (3, 0)



y-intercepts: Setting $x = 0$

$\Rightarrow y = \mathbf{0}^2 - 6(\mathbf{0}) - 6 = -6$

Therefore,

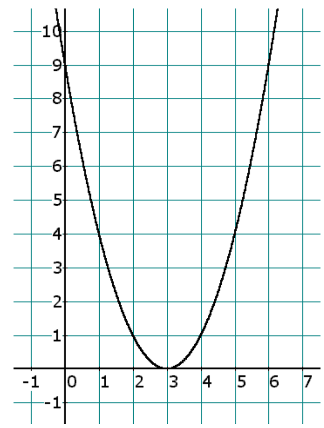
> the y-intercept is (0, –6)

EXAMPLE 4: **Find the intercepts of $y = x^2 - 6x + 9$.**

Solution:

x-intercepts: Setting $y = 0$

$$\Rightarrow \mathbf{0} = x^2 - 6x + 9$$
$$\Rightarrow 0 = (x-3)(x-3)$$
$$\Rightarrow x - 3 = 0$$
$$\Rightarrow x = 3\text{, only one solution for } x$$



We conclude that

the x-intercept is (3, 0)

y-intercepts: Setting $x = 0$

$$\Rightarrow y = (\mathbf{0})^2 - 6(\mathbf{0}) + 9 = 9$$

Therefore,

the y-intercept is (0, 9)

EXAMPLE 5: **Find the intercepts of $y = -3x^2 + 5x - 1$.**

Solution:

x-intercepts: Setting $y = 0$

$$\Rightarrow \mathbf{0} = -3x^2 + 5x - 1$$

Bringing all the terms to the left side gives us the following quadratic equation in standard form:

$$3x^2 - 5x + 1 = 0$$

The left side of the equation is not factorable (give it a try!), so we'll have to utilize either the method of completing the square or the Quadratic Formula. Let's use the Quadratic Formula, where in this case $a = 3$, $b = -5$, and $c = 1$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)}$$

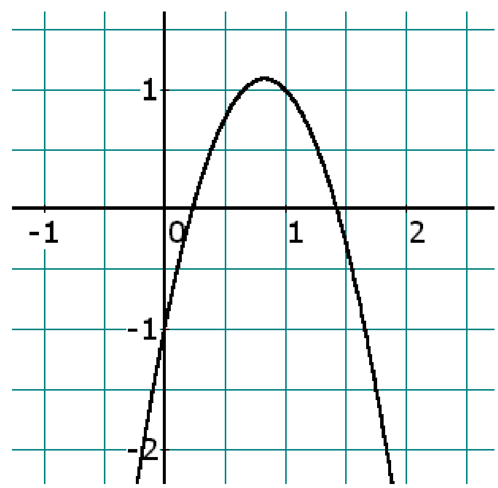
$$= \frac{5 \pm \sqrt{25 - 12}}{6} = \frac{5 \pm \sqrt{13}}{6}$$

These two solutions are certainly correct, and we could even write our two *x*-intercepts like this:

$$\left(\frac{5 + \sqrt{13}}{6}, 0\right), \quad \left(\frac{5 - \sqrt{13}}{6}, 0\right)$$

However, this form of the *x*-intercepts is not very useful for plotting them on a grid. It's better to use your calculator to convert the two exact radical answers into approximate decimal answers; our *x*-intercepts are roughly

(1.434, 0) (0.232, 0)



y-intercepts: Setting $x = 0 \Rightarrow$

$$y = -3(\mathbf{0})^2 + 5(\mathbf{0}) - 1 = -1.$$

The *y*-intercept is therefore

(0, –1)

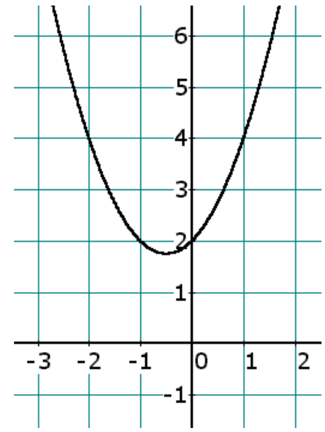
EXAMPLE 6: **Find the intercepts of $y = x^2 + x + 2$.**

Solution: Seems easy enough, but this is a strange one.

x-intercepts: Setting $y = 0$ yields the quadratic equation $x^2 + x + 2 = 0$. First, this quadratic won't factor, but that's okay; we have the Quadratic Formula to rescue us:

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2}$$

We're in trouble; the –7 in the radical sign indicates that these two solutions for x are not real numbers. This means that wherever these non-real (imaginary) numbers are, they are absolutely not on the x-axis (which is just the set of real numbers).



The conclusion of all this? This parabola has

no x-intercept

y-intercepts: Setting $x = 0$ gives $y = 2$, and so the y-intercept is

(0, 2)

Homework

4. Find all the **intercepts** of each parabola, with no approximations:

a. $y = x^2 + 2x - 48$ b. $y = x^2 + 10x + 25$

c. $y = 2x^2 + 8x + 5$ d. $y = 3x^2 - 6x + 4$

5. Find all the **intercepts** of each parabola, rounded to 3 digits:

a. $y = x^2 + 7x + 1$ b. $y = -2x^2 + 5x + 4$

c. $y = 3x^2 - 6x - 2$ d. $y = 5x^2 + 3x + 1$

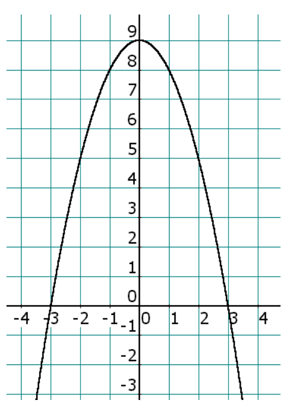
Solutions

1. The coefficient of the quadratic term in Example 1 is positive, while that of the quadratic term in Example 2 is negative. That's the clue which determines whether a parabola opens up or down. Therefore, the parabola $y = \pi x^2 - 13x + 2$ opens up, whereas the parabola $y = -0.7x^2 + 99x + 14$ opens down.

2. a. True b. False c. True d. False e. False

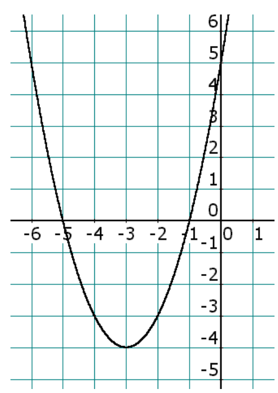
3.

a.



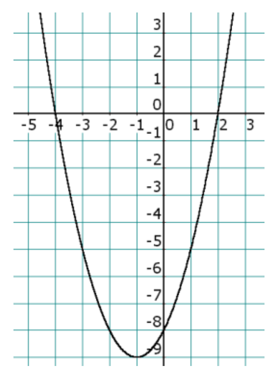
Intercepts:
(–3, 0), (3, 0), (0, 9)
Vertex: (0, 9)

b.



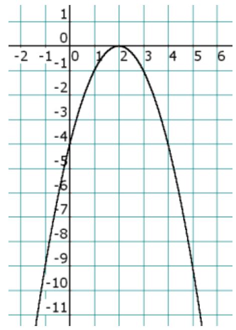
Intercepts:
(–5, 0), (–1, 0), (0, 5)
Vertex: (–3, –4)

c.



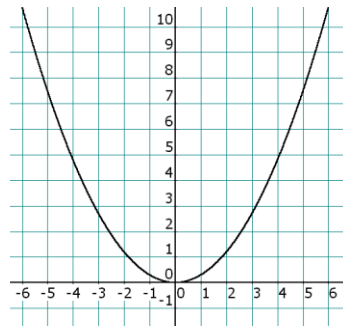
Intercepts:
(–4, 0), (2, 0), (0, –8)
Vertex: (–1, –9)

d.



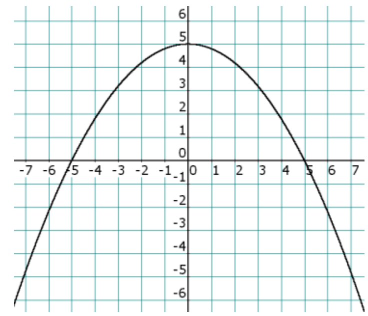
Intercepts:
(2, 0), (0, –4)
Vertex: (2, 0)

e.



Intercepts: (0, 0)
Vertex: (0, 0)

f.



Intercepts:
(–5, 0), (5, 0), (0, 5)
Vertex: (0, 5)

4. a. (–8, 0) (6, 0) (0, –48)

b. (–5, 0) (0, 25)

c. $\left(\dfrac{-4+\sqrt{6}}{2},\ 0\right)$ $\left(\dfrac{-4-\sqrt{6}}{2},\ 0\right)$ (0, 5)

d. No x-intercepts (0, 4)

5. a. (–0.146, 0) (–6.854, 0) (0, 1)
b. (3.137, 0) (–0.637, 0) (0, 4)
c. (2.291, 0) (–0.291, 0) (0, –2)
d. No x-intercepts (0, 1)

“It is our choices that show what we truly are, far more than our abilities.”

– *J.K. Rowling*